FUZZY STATISTICS

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From Crisp Sets to Fuzzy Sets

Chapter 1

1.1 INTRODUCTION

UNCERTAINTY

Uncertainties arise from many sources:

❑ Random effects

❑ Measurement errors

❑ Modeling choices

❑ Parameter choices

❑ Inference processes

❑ Decision making

❑ lack of knowledge

1.1 INTRODUCTION

UNCERTAINTY TYPES

Three types of uncertainty are now recognized in the five theories:

- ❑ Nonspecificity (or imprecision)

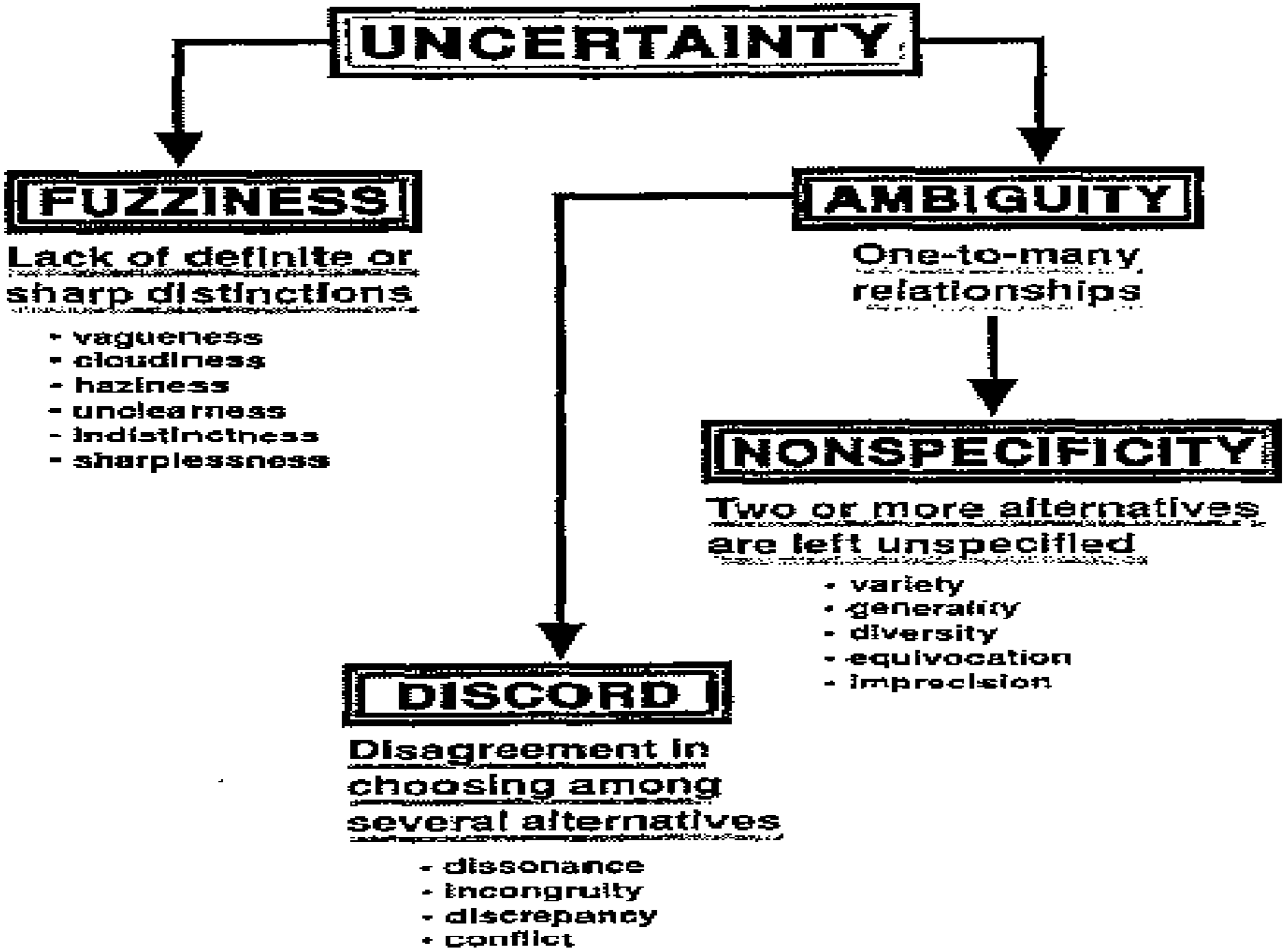
which is connected with sizes (cardinalities) of relevant sets of alternatives.

- ❑ Fuzziness (or vagueness),

which results from imprecise boundaries of fuzzy sets.

- ❑ Strife (or discord)

which expresses conflicts among the various sets of alternatives.


UNCERTAINTY
FUZZINESS
Lack of definite or sharp distinctions
- vagueness
- cloudiness
- haziness
- unclearness
- indistinctness
- sharplessness
AMBIGUITY
One-to-many relationships
NONSPECIFICITY
Two or more alternatives are left unspecified
- variety
- generality
- diversity
- equivocation
- imprecision
DISCORD
Disagreement in choosing among several alternatives
- dissonance
- incongruity
- discrepancy
- conflict

1.1 INTRODUCTION

Crisp Sets versus Fuzzy Sets

- The crisp set is defined in such a way as to dichotomize the individuals in some given universe of discourse into two groups: members (those that certainly belong in the set) and nonmembers (those that certainly, do not).
- A fuzzy set can be defined mathematically by assigning to each possible individual in the universe of discourse a value representing its grade of membership in the fuzzy set.

1.1 INTRODUCTION

FROM CRISP SETS TO FUZZY SETS

- Probability theory is capable of representing only one of several distinct types of uncertainty.
- When A is a fuzzy set and x is a relevant object, the proposition "x is a member of A" is not necessarily either true or false. It may be true only to some degree , the degree to which x is actually a member of A.

1.2 CRISP SETS: AN OVERVIEW

The theory of crisp set

- The following general symbols are employed throughout the text:

$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ (the set of all integers),

$\mathbb{N} = \{1, 2, 3, \ldots\}$ (the set of all positive integers or natural numbers),

$\mathbb{N}_0 = \{0, 1, 2, \ldots\}$ (the set of all nonnegative integers),

$\mathbb{N}_n = \{1, 2, \ldots, n\}$,

$\mathbb{N}_{0,n} = \{0, 1, \ldots, n\}$,

$\mathbb{R}$: the set of all real numbers,

$\mathbb{R}^+$: the set of all nonnegative real numbers,

$[a, b], (a, b], [a, b), (a, b)$: closed, left-open, right-open, open interval of real numbers between a and b, respectively,

$\langle x_1, x_2, \ldots, x_n \rangle$: ordered n-tuple of elements $x_1, x_2, \ldots, x_n$.

1.2 CRISP SETS: AN OVERVIEW

Three basic methods to define sets:

- The list method: a set is defined by naming all its members.

$$A = \{a_1, a_2, ..., a_n\}$$

- The rule method: a set is defined by a property satisfied by its members.

$$A = \{x \mid P(x)\}$$

where '|' denotes the phrase "such that"

$P(x)$: a proposition of the form "x has the property P "

- A set is defined by a characteristic function.

$$\chi_A(x) = \begin{cases} 1 & \text{for } x \in A \\ 0 & \text{for } x \notin A \end{cases}$$

the characteristic function $\chi_A : X \to \{0,1\}$

1.2 CRISP SETS: AN OVERVIEW

The union of sets A and B:

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

The generalized union operation: for a family of sets,

$$\bigcup_{i \in I} A_i = \{x \mid x \in A_i \text{ for some } i \in I\}$$

The intersection of sets A and B:

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

The generalized intersection operation: for a family of sets,

$$\bigcap_{i \in I} A_i = \{x \mid x \in A_i \text{ for all } i \in I\}$$

1.2 CRISP SETS: AN OVERVIEW

TABLE 1.1 FUNDAMENTAL PROPERTIES OF CRISP SET OPERATIONS

Involution	$\overline{\overline{A}} = A$
Commutativity	$A \cup B = B \cup A$
	$A \cap B = B \cap A$
Associativity	$(A \cup B) \cup C = A \cup (B \cup C)$
	$(A \cap B) \cap C = A \cap (B \cap C)$
Distributivity	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
Idempotence	$A \cup A = A$
	$A \cap A = A$
Absorption	$A \cup (A \cap B) = A$
	$A \cap (A \cup B) = A$
Absorption by X and $\varnothing$	$A \cup X = X$
	$A \cap \varnothing = \varnothing$
Identity	$A \cup \varnothing = A$
	$A \cap X = A$
Law of contradiction	$A \cap \overline{A} = \varnothing$
Law of excluded middle	$A \cup \overline{A} = X$
De Morgan's laws	$\overline{A \cap B} = \overline{A} \cup \overline{B}$
	$\overline{A \cup B} = \overline{A} \cap \overline{B}$

1.2 CRISP SETS: AN OVERVIEW

Let R denote a set of real number.

- If there is a real number r such that $x \leq r$ for every $x \in R$, then r is called an upper bound of R, and R is bounded above by r.
- If there is a real number s such that $x \geq s$ for every $x \in R$, then s is called an lower bound of R, and R is bounded below by s.

For any set of real numbers R that is bounded above, a real number r is called the supremum of R (write $r = \sup R$) iff

(a) r is an upper bound of R;

(b) no number less than r is an upper bound of R.

For any set of real numbers R that is bounded below, a real number s is called the infimum of R (write $s = \inf R$) iff

(a) s is an lower bound of R;

(b) no number greater than s is an lower bound of R.

1.3 FUZZY SETS: BASIC TYPES

A membership function:

- A characteristic function: the values assigned to the elements of the universal set fall within a specified range and indicate the membership grade of these elements in the set.
- Larger values denote higher degrees of set membership.

A set defined by membership functions is a fuzzy set.

The most commonly used range of values of membership functions is the unit interval [0,1].

We think the universal set X is always a crisp set.

Notation:

- The membership function of a fuzzy set A is denoted by μ_A :

$$\mu_A : X \to [0,1]$$

- In the other one, the function is denoted by A and has the same form

$$A : X \to [0,1]$$

1.3 FUZZY SETS: BASIC TYPES

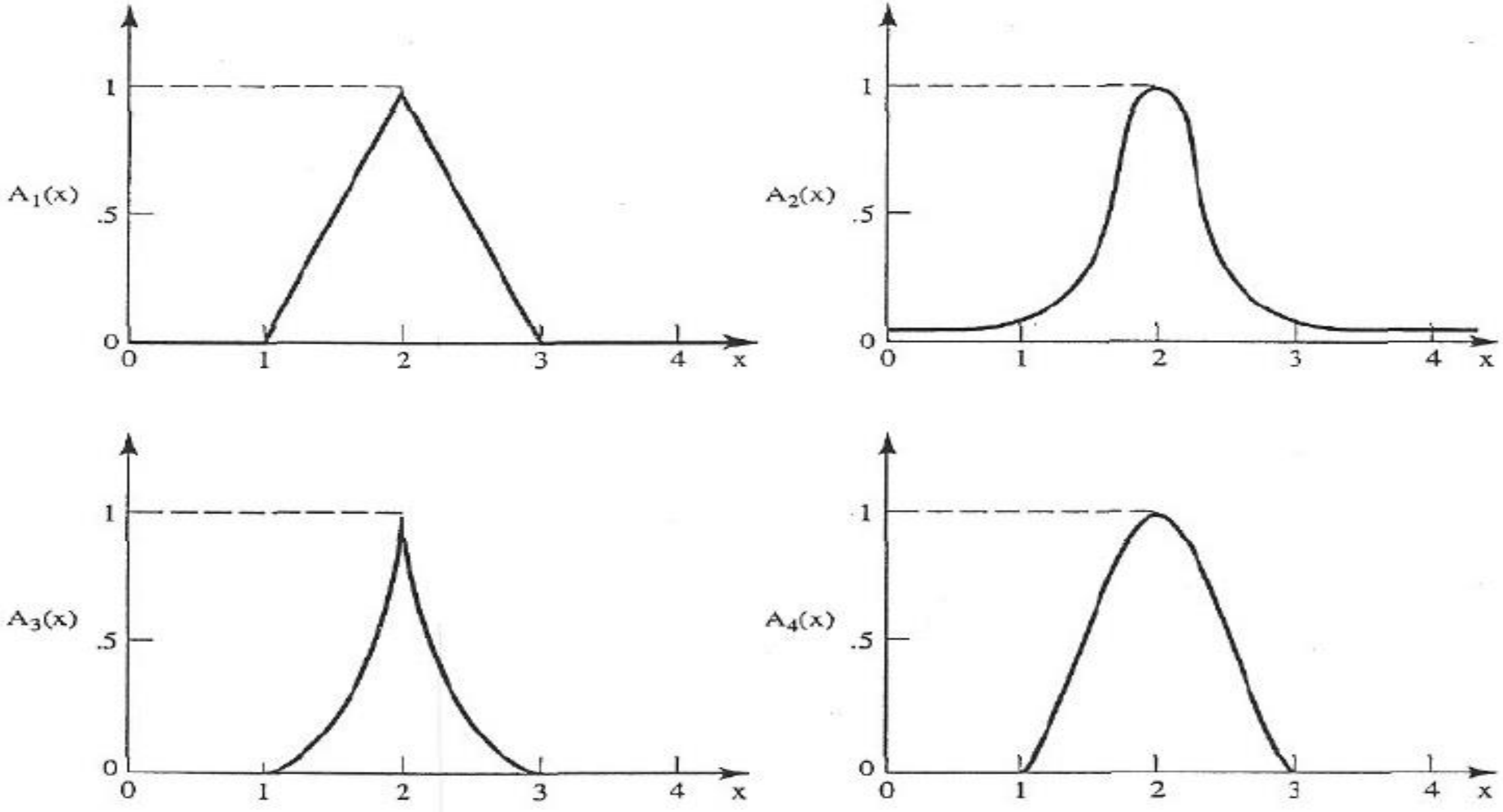


Figure 1.2 Examples of membership functions that may be used in different contexts for characterizing fuzzy sets of real numbers close to 2.

1.3 FUZZY SETS: BASIC TYPES

The four fuzzy sets are similar in the sense that the following properties are possessed by each $A_i (i \in \mathbb{N}_4)$:

(i) $A_i(2) = 1$ and $A_i(x) < 1$ for all $x \neq 2$;
(ii) A_i is symmetric with respect to $x = 2$, that is $A_i(2 + x) = A_i(2 - x)$ for all $x \in \mathbb{R}$;
(iii) $A_i(x)$ decreases monotonically from 1 to 0 with the increasing difference $|2 - x|$.

Each function in Fig. 1.2 is a member of a parameterized family of functions.

$$A_1(x) = \begin{cases} p_1(x - r) + 1 & \text{when } x \in [r - 1/p_1, r] \\ p_1(r - x) + 1 & \text{when } x \in [r, r + 1/p_1] \\ 0 & \text{otherwise} \end{cases}$$

$$A_2(x) = \frac{1}{1 + p_2(x - r)^2}$$

$$A_3(x) = e^{-|p_3(x-r)|}$$

$$A_4(x) = \begin{cases} (1 + \cos(p_4\pi(x - r)))/2 & \text{when } x \in [r - 1/p_4, r + 1/p_4] \\ 0 & \text{otherwise} \end{cases}$$

1.3 FUZZY SETS: BASIC TYPES

An example:

- Define the seven levels of education:

0 – no education
1 – elementary school
2 – high school
3 – two-year college degree
4 – bachelor's degree
5 – master's degree
6 – doctoral degree

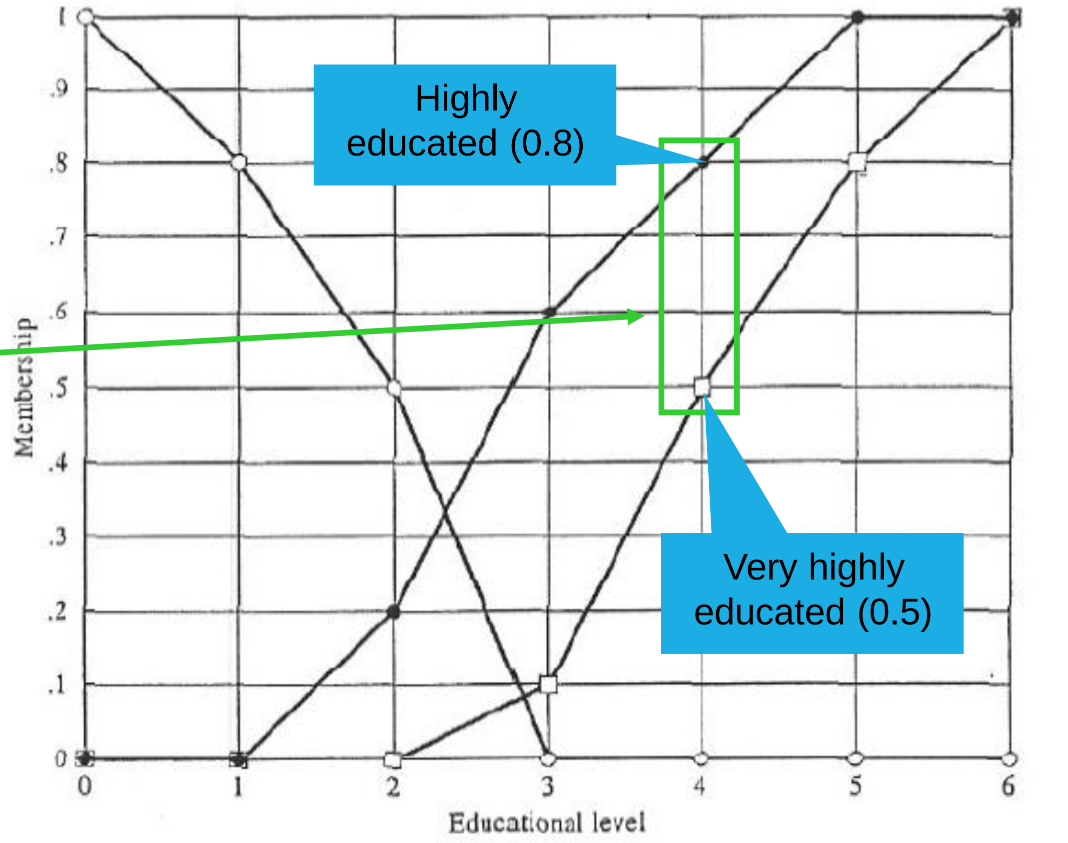


Figure 1.3 Examples of fuzzy sets expressing the concepts of people that are little educated (○), highly educated (●), and very highly educated (□).

1.3 FUZZY SETS: BASIC TYPES

Several fuzzy sets representing linguistic (Value) concepts such as low, medium, high, and so one are often employed to define states of a variable. Such a variable is usually called a fuzzy variable.

For example:

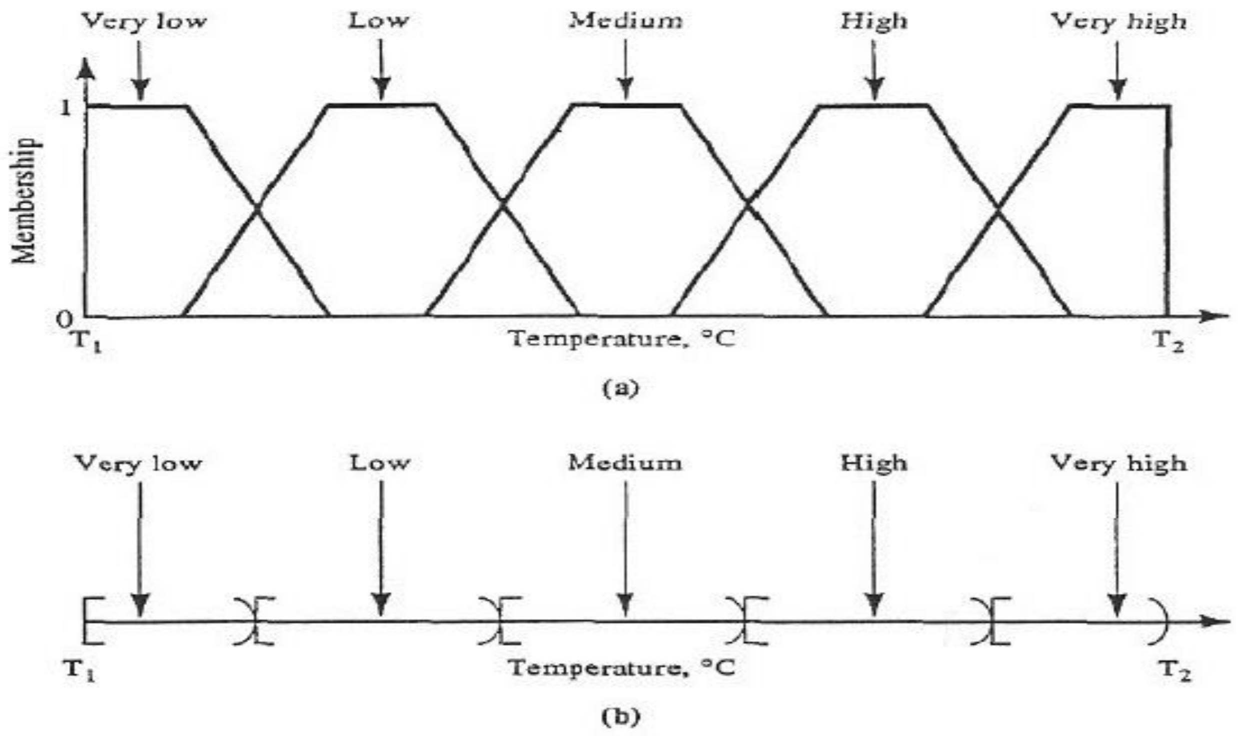
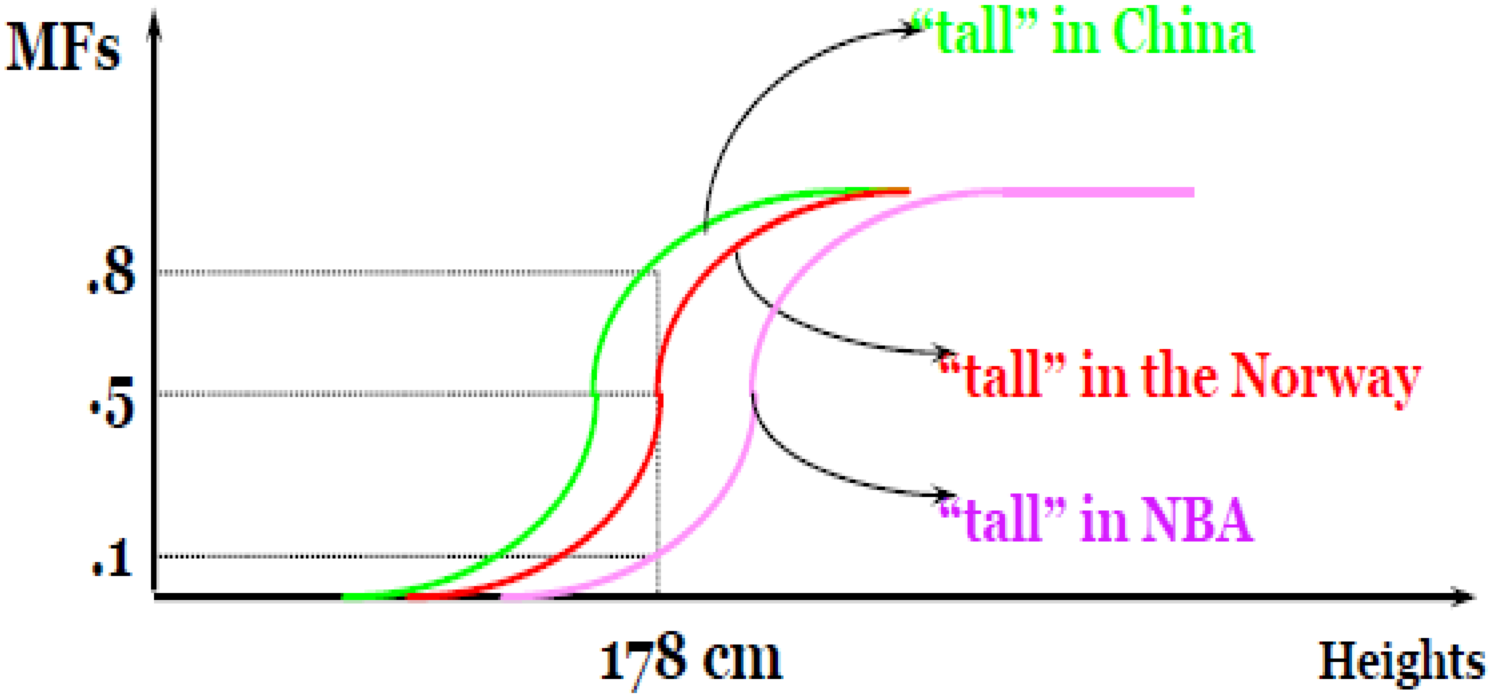


Figure 1.4 Temperature in the range $[T_1, T_2]$ conceived as: (a) a fuzzy variable; (b) a traditional (crisp) variable.

1.3 FUZZY SETS: MEMBERSHIP FUNCTIONS (MFS)

- Characteristics of MFs:
  - Subjective measures
  - Not probability functions



1.3 FUZZY SETS: MEMBERSHIP FUNCTIONS (MFS)

- **Formal definition:**
  A fuzzy set *A* in *X* is expressed as a set of ordered pairs:

$$A = \{(x, \mu_A(x)) | x \in X\}$$

Fuzzy set | **Membership function (MF)** | **X: Universe or universe of discourse**

A fuzzy set is totally characterized by a membership function (MF).

1.3 FUZZY SETS: FUZZY SETS WITH DISCRETE UNIVERSES

- Fuzzy set **C** = “desirable city to live in”

X = {Baghdad, Basra, Erbil} (**discrete** and **non-ordered**)

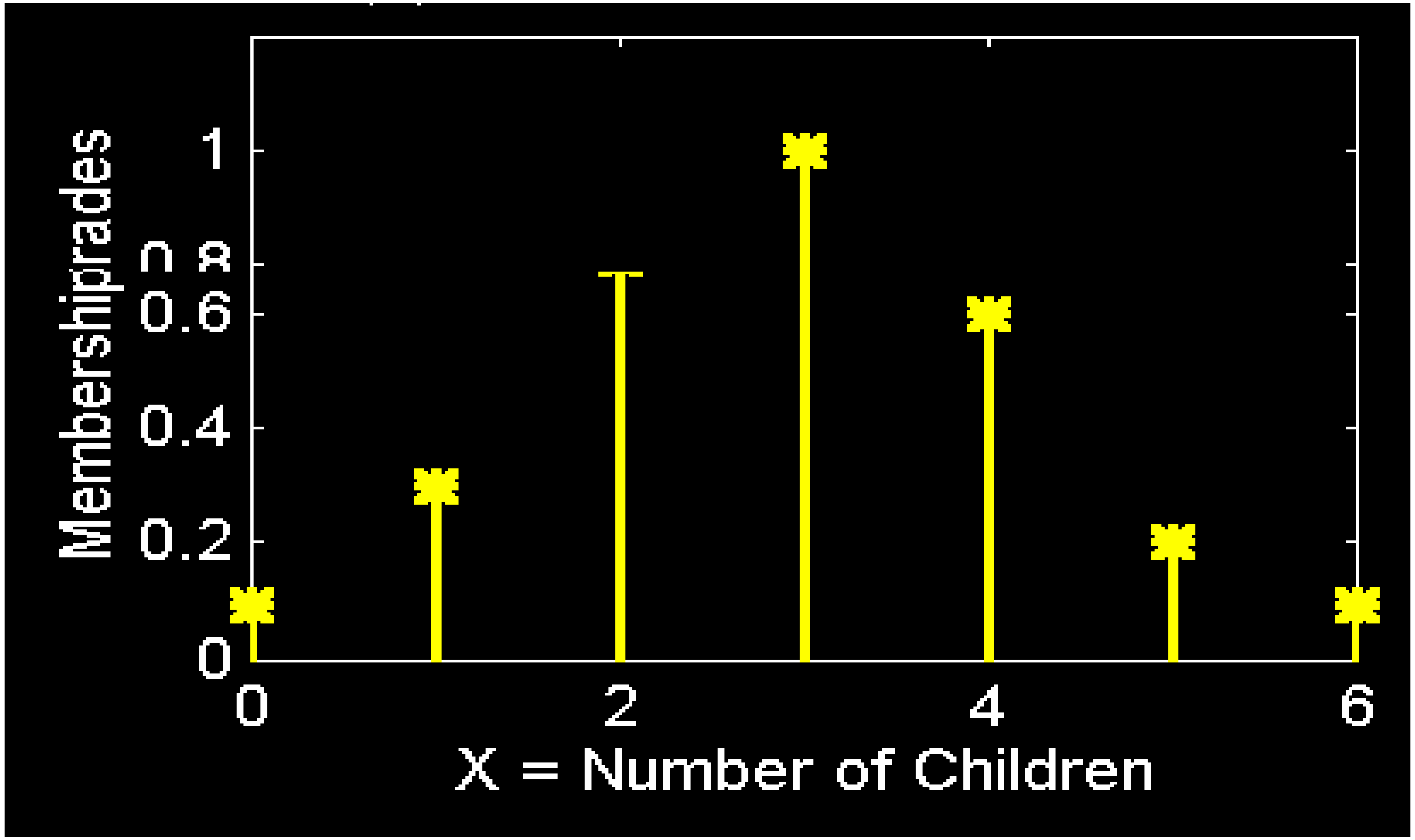
C = {(Baghdad, 0.1), (Basra, 0.8), (Erbil, 0.9)}

- Fuzzy set **A** = “sensible number of children in a family”

X = {0, 1, 2, 3, 4, 5, 6} (**discrete ordered** universe)

A = {(0, .1), (1, .3), (2, .7), (3, 1), (4, .6), (5, .2), (6, .1)}

1.3 FUZZY SETS: FUZZY SETS WITH DISCRETE UNIVERSES



A = {(0, .1), (1, .3), (2, .7), (3, 1), (4, .6), (5, .2), (6, .1)}

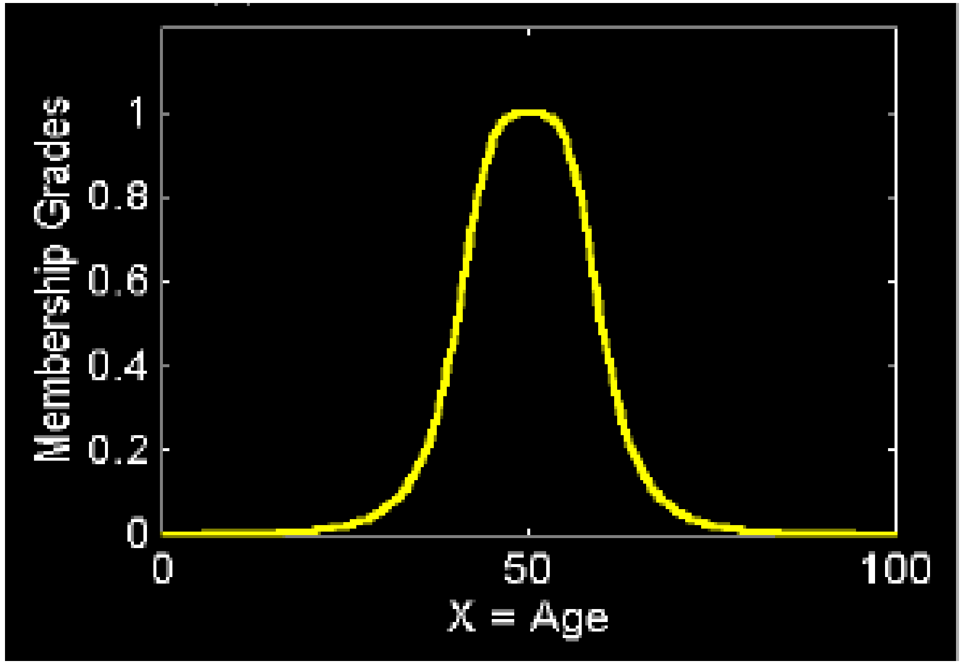
1.3 FUZZY SETS: FUZZY SETS WITH CONT. UNIVERSES

Fuzzy set **B** = “**about 50 years old**”

X = Set of positive real numbers (**continuous**)

B = {(x, μ_B(x)) | x in X}

$$\mu_B(x) = \frac{1}{1 + \left(\frac{x - 50}{10}\right)^2}$$



1.3 FUZZY SETS: ALTERNATIVE NOTATION

- A fuzzy set A can be alternatively denoted as follows:

X is discrete $\Rightarrow$ $$A = \sum_{x_i \in X} \mu_A(x_i) / x_i$$

X is continuous $\Rightarrow$ $$A = \int_X \mu_A(x) / x$$

Note that ∑ and integral signs stand for the union of membership grades; "/" stands for a marker and does not imply division.

1.3 FUZZY SETS: ALTERNATIVE NOTATION

- Examples:

A is discrete

$$A = \{(0, 0.1), (1, 0.3), (2, 0.7), (3, 1), (4, 0.7), (5, 0.3), (6, 0.1)\}.$$

$$A = 0.1/0 + 0.3/1 + 0.7/2 + 1.0/3 + 0.7/4 + 0.3/5 + 0.1/6,$$

C is discrete

$$C = \{(\text{San Francisco}, 0.9), (\text{Boston}, 0.8), (\text{Los Angeles}, 0.6)\}.$$

$$C = 0.9/\text{San Francisco} + 0.8/\text{Boston} + 0.6/\text{Los Angeles},$$

B is continuous

$$B = \text{“about 50 years old”} \qquad \mu_B(x) = \frac{1}{1 + \left(\frac{x-50}{10}\right)^4}.$$

$$B = \{(x, \mu_B(x)) | x \in X\},$$

$$B = \int_{R^+} \frac{1}{1 + (\frac{x-50}{10})^4} \Big/ x,$$

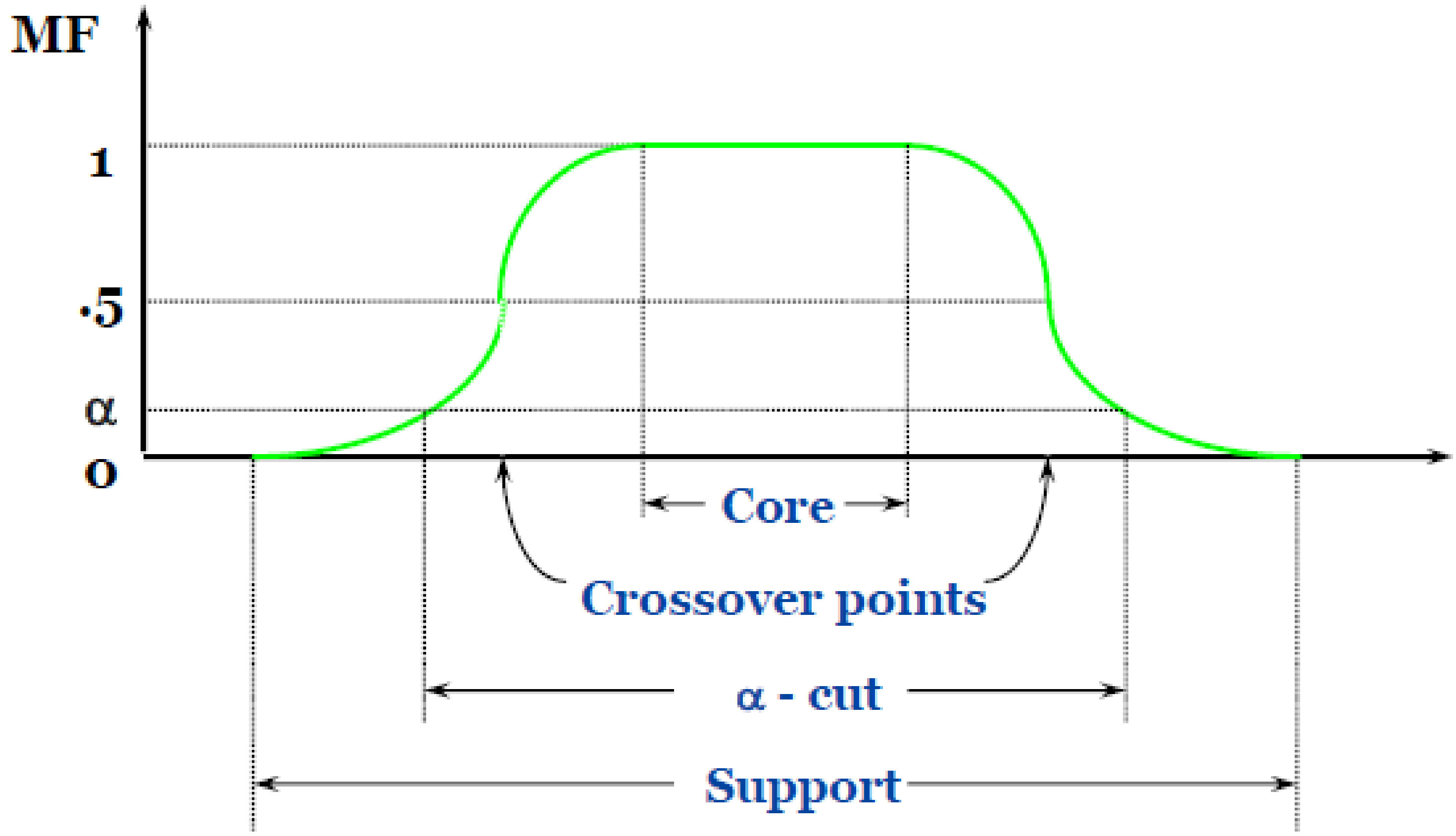
1.3 FUZZY SETS: MEMBERSHIP FUNCTIONS?

- Subjective evaluation: The shape of the functions is defined by specialists
- Ad-hoc: choose a simple function that is suitable to solve the problem
- Distributions, probabilities: information extracted from measurements
- Adaptation: testing
- Automatic: algorithms used to define functions from data

1.3 FUZZY SETS: MEMBERSHIP FUNCTIONS TERMINOLOGY

Some Definitions

- Support
- Core
- Crossover points (equilibrium points)
- α-cut, strong α -cut



1.3 FUZZY SETS: MEMBERSHIP FUNCTIONS TERMINOLOGY

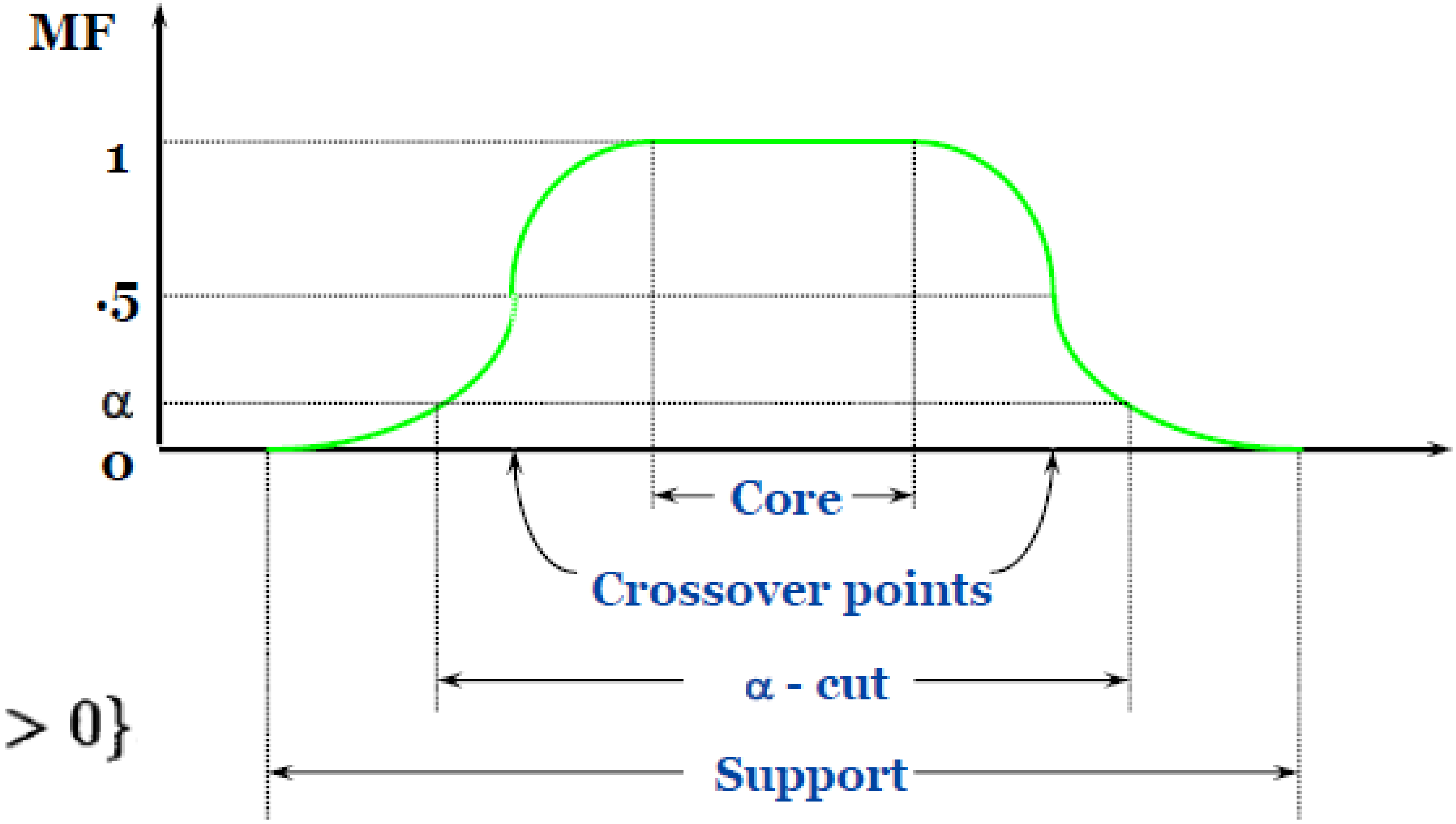
Support

- The Support of a fuzzy set **A** is the set of all points x in **X** such that:

$$\mu_A(x) > 0$$

- In other words:

$$\text{support}(A) = \{x | \mu_A(x) > 0\}$$



1.3 FUZZY SETS: MEMBERSHIP FUNCTIONS TERMINOLOGY

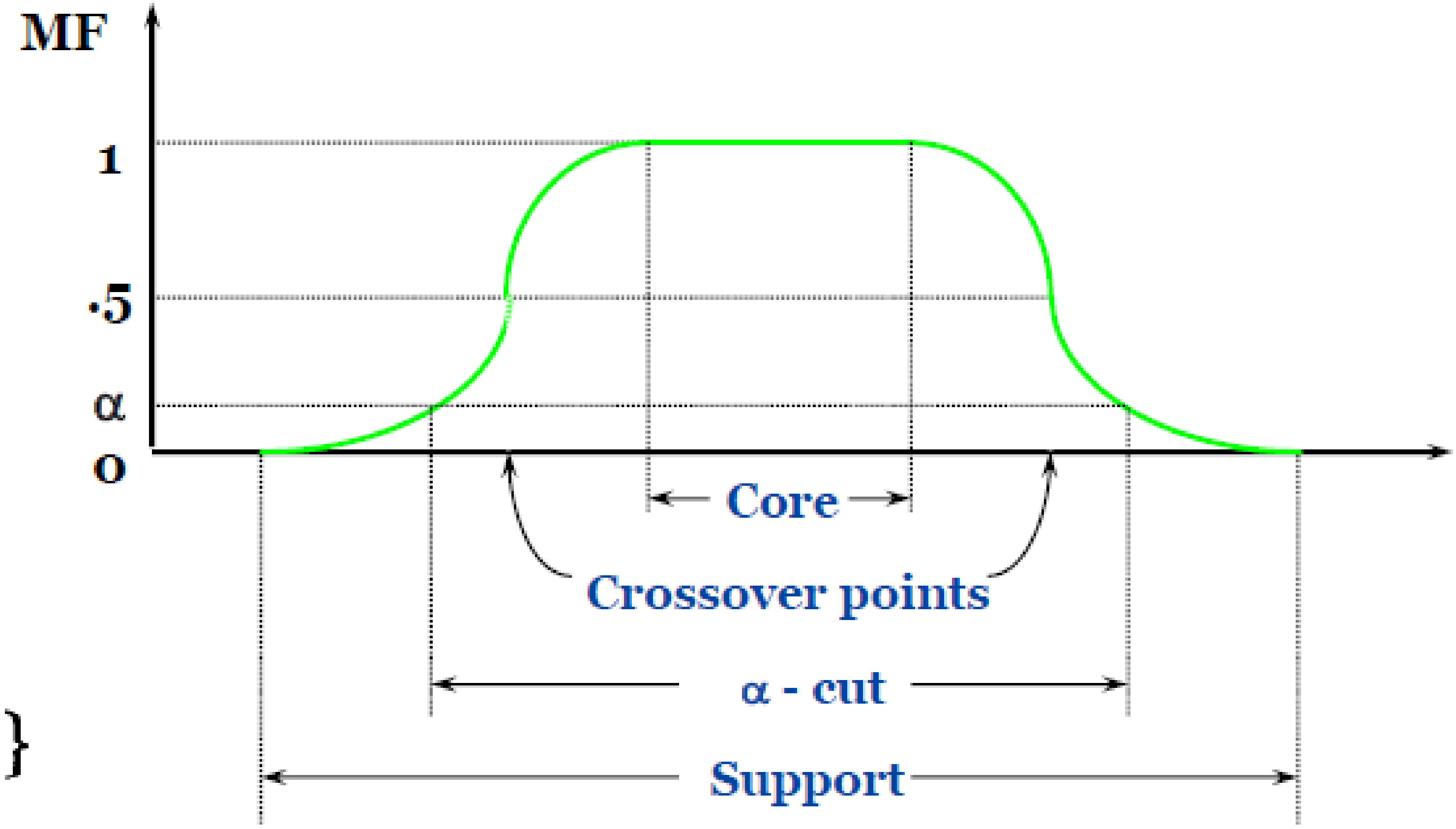
Core

- The Core of a fuzzy set **A** is the set of all points x in **X** such that:

$$\mu_A(x) = 1$$

- In other words:

$$\text{core}(A) = \{x \mid \mu_A(x) = 1\}$$



1.3 FUZZY SETS: MEMBERSHIP FUNCTIONS TERMINOLOGY

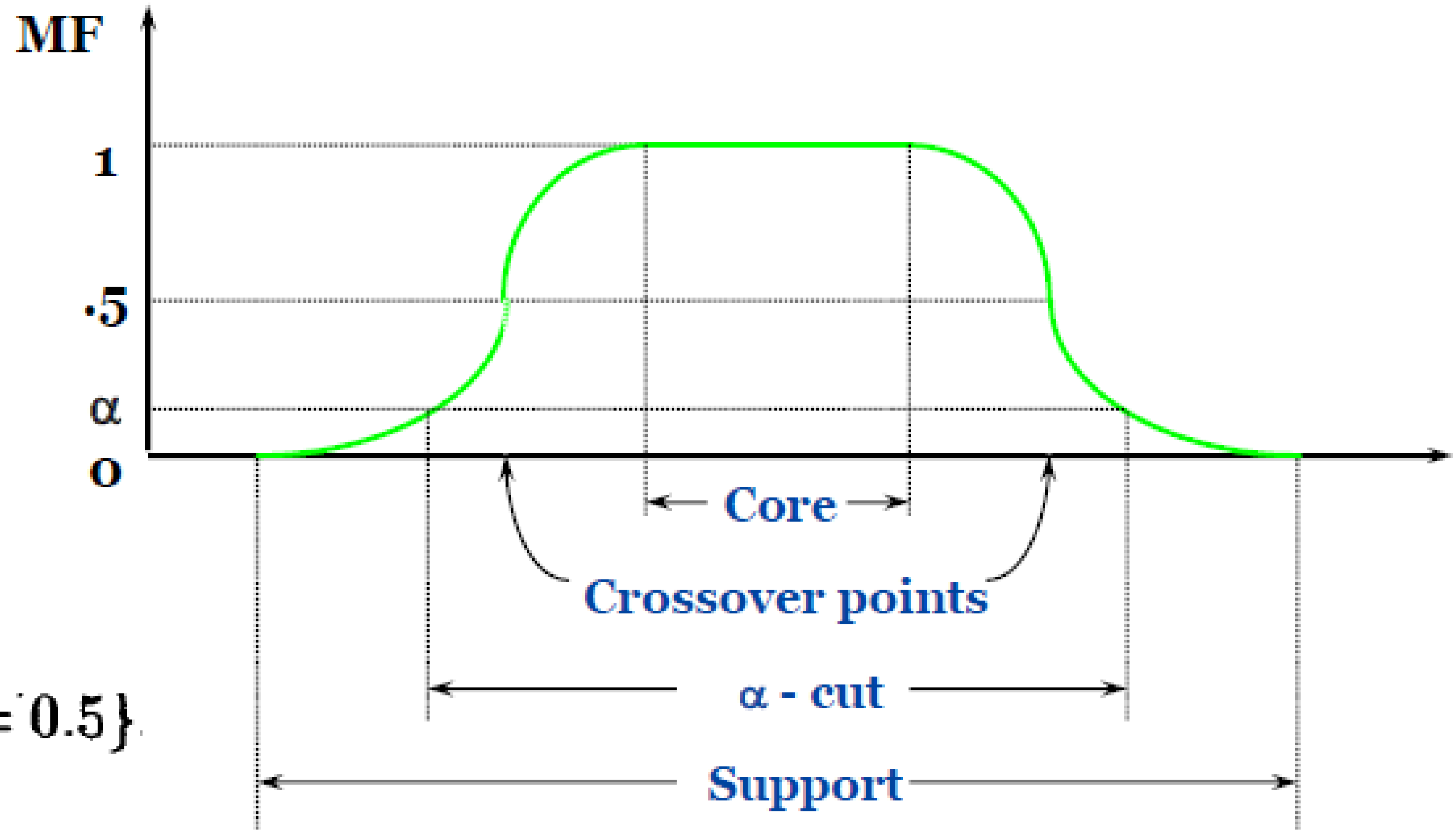
Crossover point

- The Crossover point of a fuzzy set **A** is the set of all points x in **X** such that:

$$\mu_A(x) = 0.5$$

- In other words:

$$\text{crossover}(A) = \{x \mid \mu_A(x) = 0.5\}$$

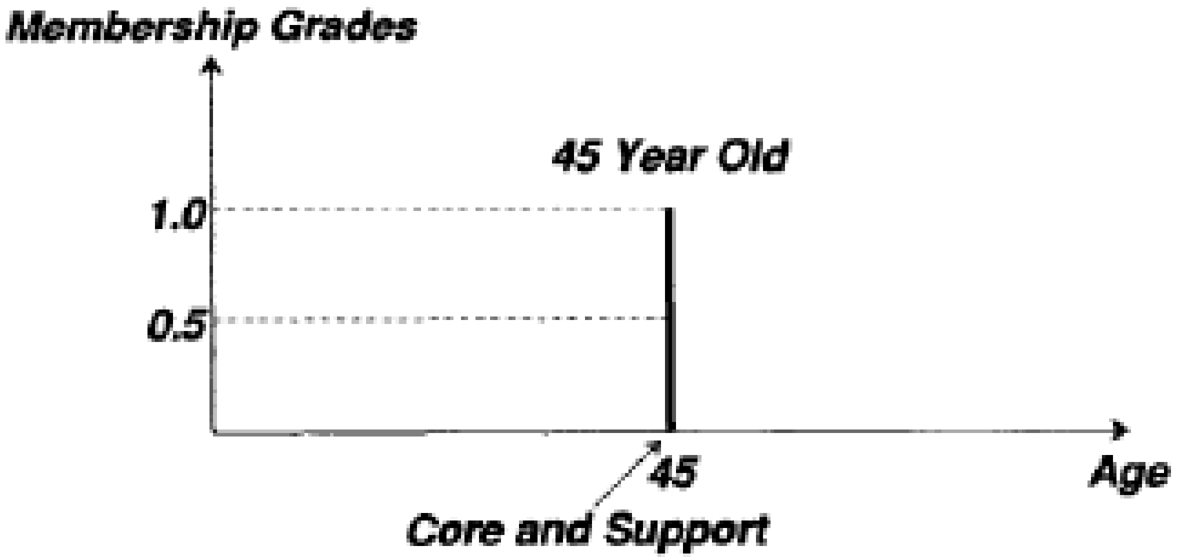


1.3 FUZZY SETS: MEMBERSHIP FUNCTIONS TERMINOLOGY

Fuzzy Singleton

- A fuzzy set whose support is a single point in **X** is a fuzz singleton if: $\mu_A(x) = 1$

- Example:
- A fuzzy singleton
- "45 years old"



1.3 FUZZY SETS: MEMBERSHIP FUNCTIONS TERMINOLOGY

α -cut and strong α -cut

- Given a fuzzy set A defined on X and any number $\alpha \in [0,1]$,

  the α -cut and strong α-cut are the crisp sets:

$$^{\alpha}A = \{x | A(x) \geq \alpha\}$$

$$^{\alpha+}A = \{x | A(x) > \alpha\}.$$

- The α -cut of a fuzzy set A is the crisp set that contains all the elements of the universal set X whose membership grades in A are greater than or equal to the specified value of α .
- The strong α -cut of a fuzzy set A is the crisp set that contains all the elements of the universal set X whose membership grades in A are only greater than the specified value of α .

1.3 FUZZY SETS: MEMBERSHIP FUNCTIONS TERMINOLOGY

The height of a fuzzy set A:

- The height of a fuzzy set A is the largest membership grade obtained by any element in that set.

$$h(A) = \sup_{x \in X} A(x)$$

- A fuzzy set A is called normal when $h(A) = 1$.
- It is called subnormal when $h(A) < 1$.
- The height of A may also be viewed as the supremum of α for which ${}^{\alpha}A \neq \phi$.

1.3 FUZZY SETS: MEMBERSHIP FUNCTIONS TERMINOLOGY

- Scalar cardinality: The cardinality of a fuzzy set is equal

to the sum of the membership degrees of all elements.

- The cardinality is represented by |A|

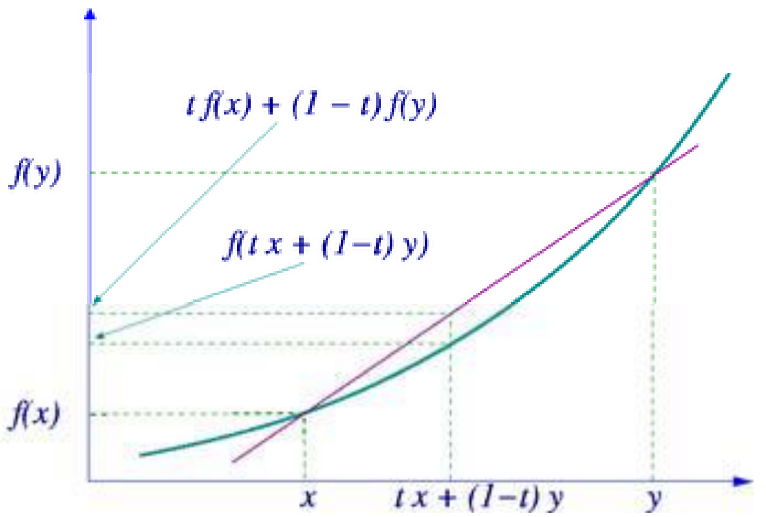
$$|A| = \sum_{i=1}^{n} \mu_A(x_i)$$

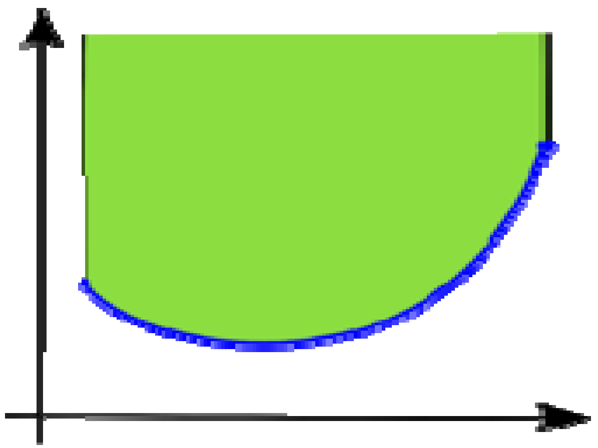
1.3 FUZZY SETS: CONCAVE FUNCTION

In mathematics, a real-valued function f defined on an interval is called convex, if for any two points x and y in its domain C and any t in [0,1], we have

$$f(tx + (1-t)y) \leq t f(x) + (1-t) f(y).$$

- In other words, a function is convex if and only if its epigraph (the set of points lying on or above the graph) is a convex set.



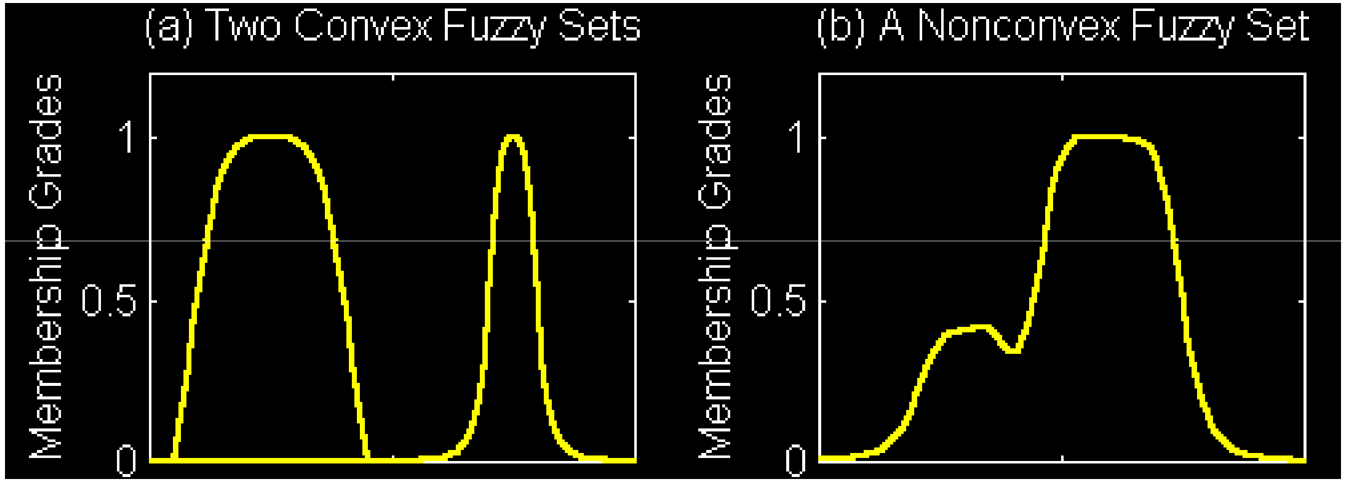


1.3 FUZZY SETS: CONVEXITY OF FUZZY SETS

- A fuzzy set A is convex if for any α in [0, 1],

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu_A(x_1), \mu_A(x_2))$$

Alternatively, A is convex is all its α-cuts are convex.



1.4 FUZZY SETS: THEORETIC OPERATIONS COMPLEMENT

The standard complement of fuzzy set A with respect to the universal set X is defined for all $x \in X$ by the equation $\overline{A}(x) = 1 - A(x)$

Elements of X for which $\overline{A}(x) = A(x)$ are called equilibrium points of A.

For example, the equilibrium points of A2 in Fig. 1.7 are 27.5 and 52.5.

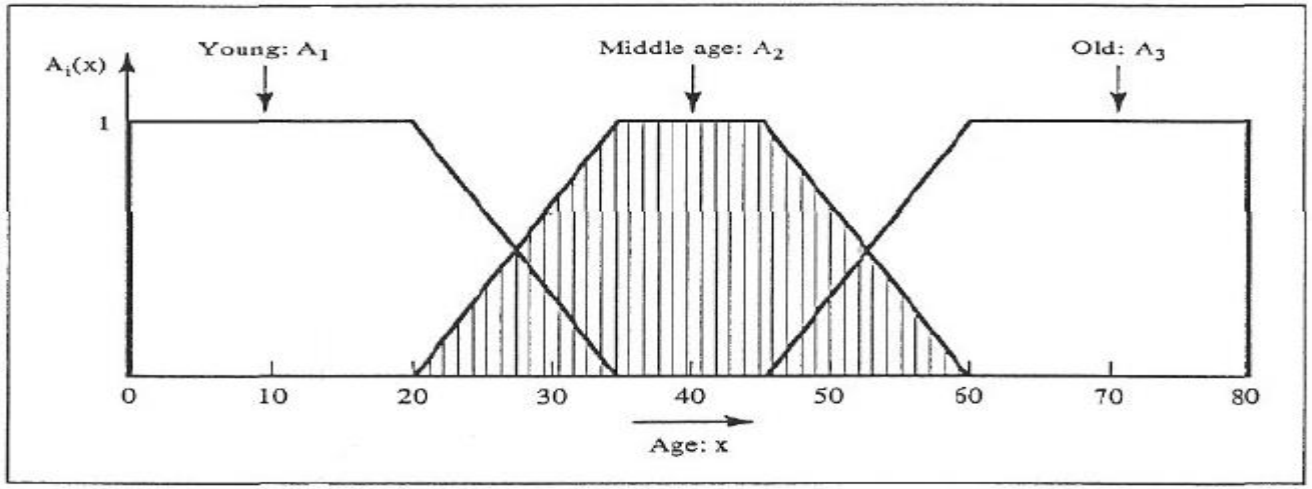
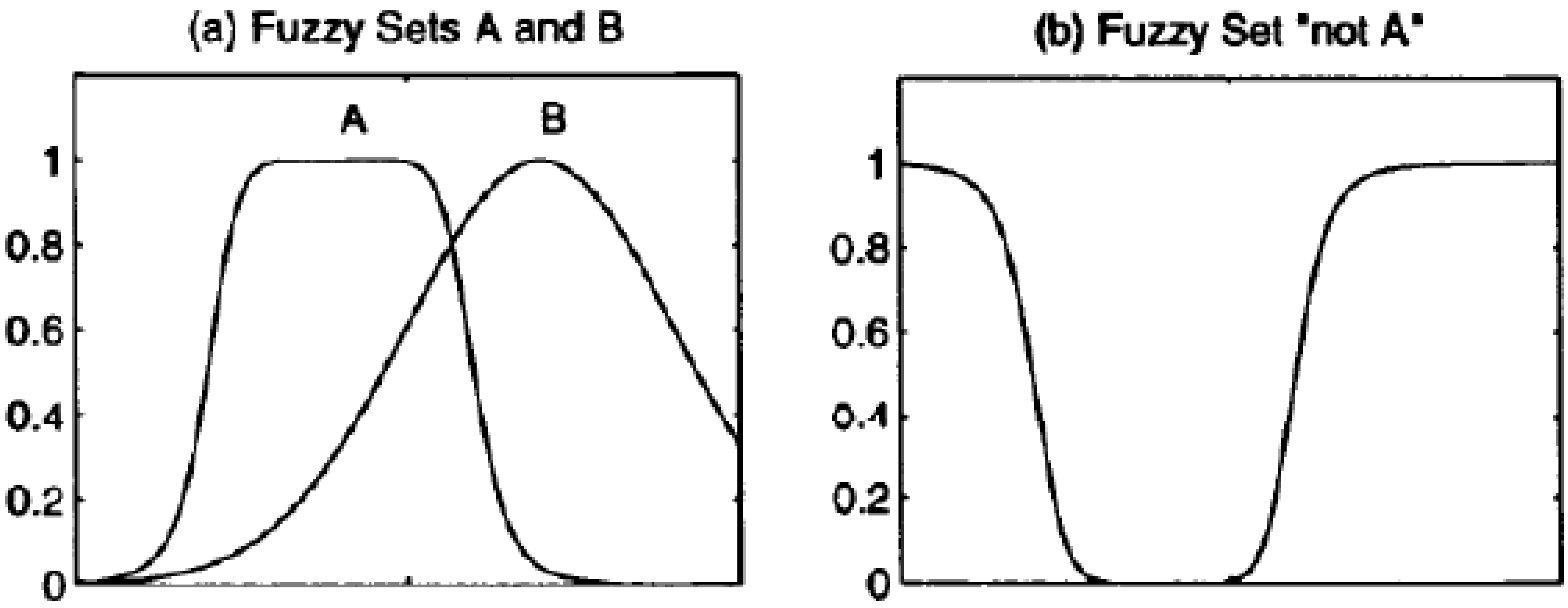


Figure 1.7 Membership functions representing the concepts of a young, middle-aged, and old person. Shown discrete approximation D_2 of A_2 is defined numerically in Table 1.2.

1.4 FUZZY SETS: THEORETIC OPERATIONS COMPLEMENT



1.4 FUZZY SETS: THEORETIC OPERATIONS INTERSECTION AND UNION

Given two fuzzy sets, A and B, their intersection and union are defined for all $x \in X$ by the equations

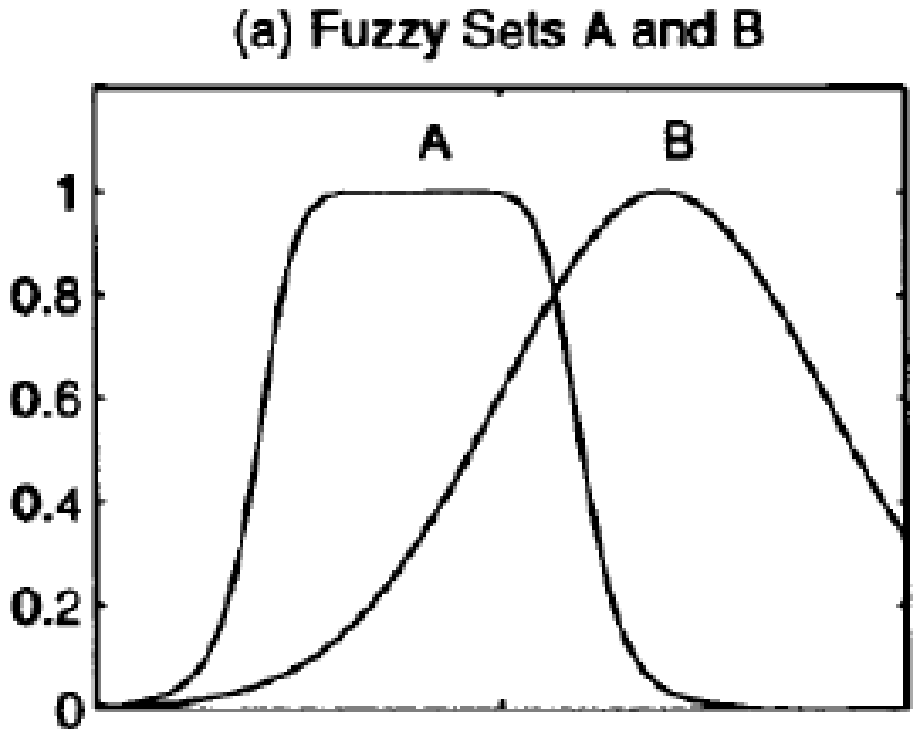
$$(A \cap B)(x) = \min[A(x), B(x)],$$

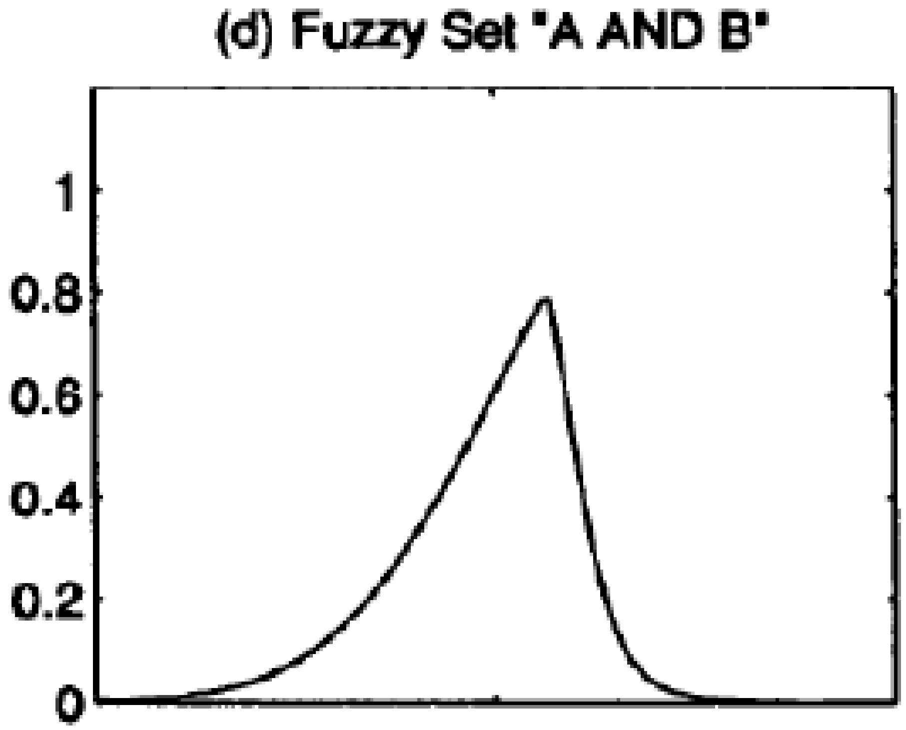
$$(A \cup B)(x) = \max[A(x), B(x)],$$

$$C = A \cap B \Leftrightarrow \mu_c(x) = \min(\mu_A(x), \mu_B(x)) = \mu_A(x) \wedge \mu_B(x)$$

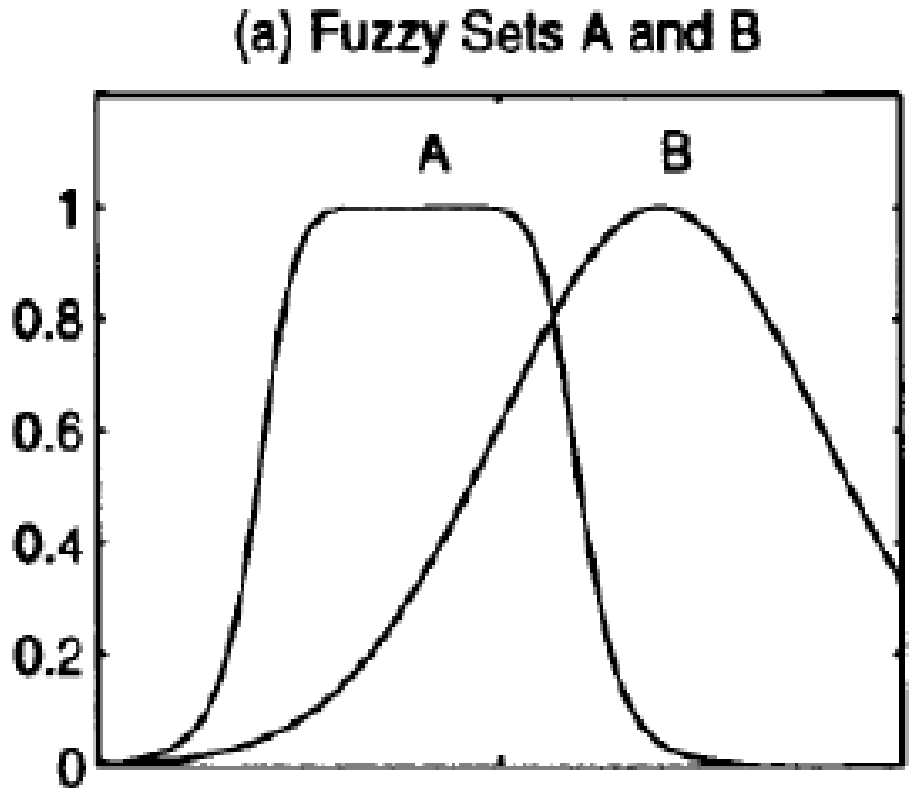
$$C = A \cup B \Leftrightarrow \mu_c(x) = \max(\mu_A(x), \mu_B(x)) = \mu_A(x) \vee \mu_B(x)$$

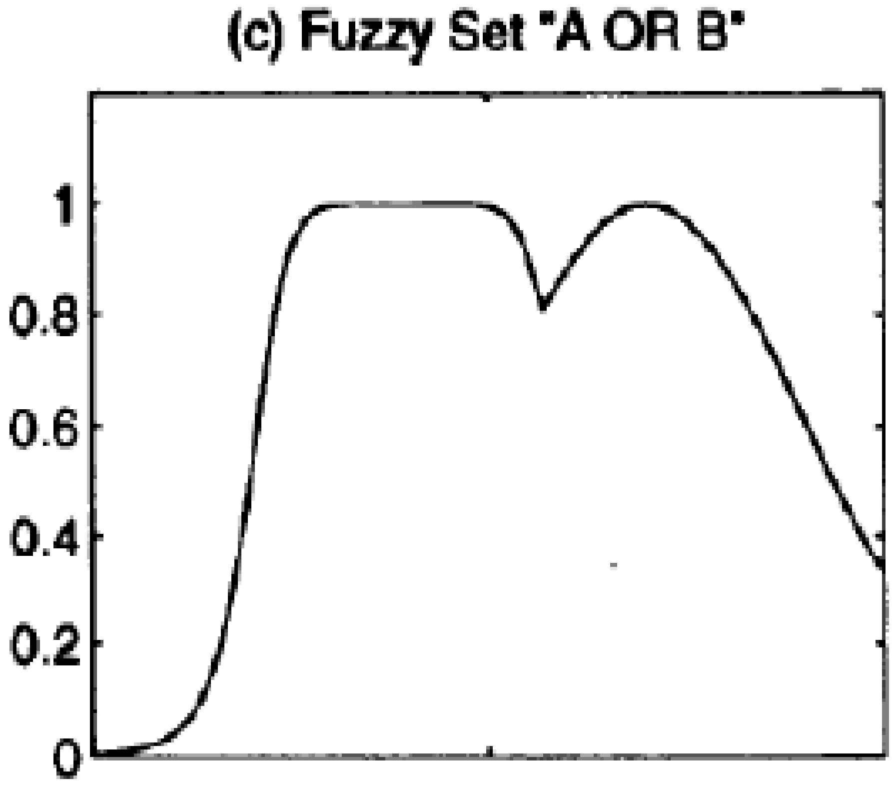
1.4 FUZZY SETS: THEORETIC OPERATIONS INTERSECTION





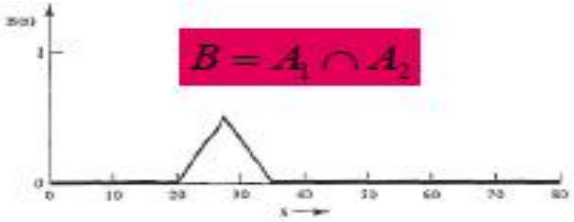
1.4 FUZZY SETS: THEORETIC OPERATIONS UNION

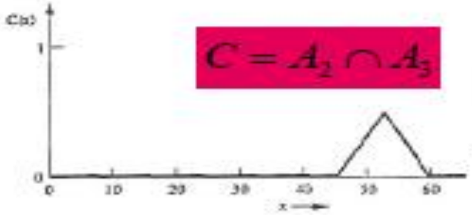




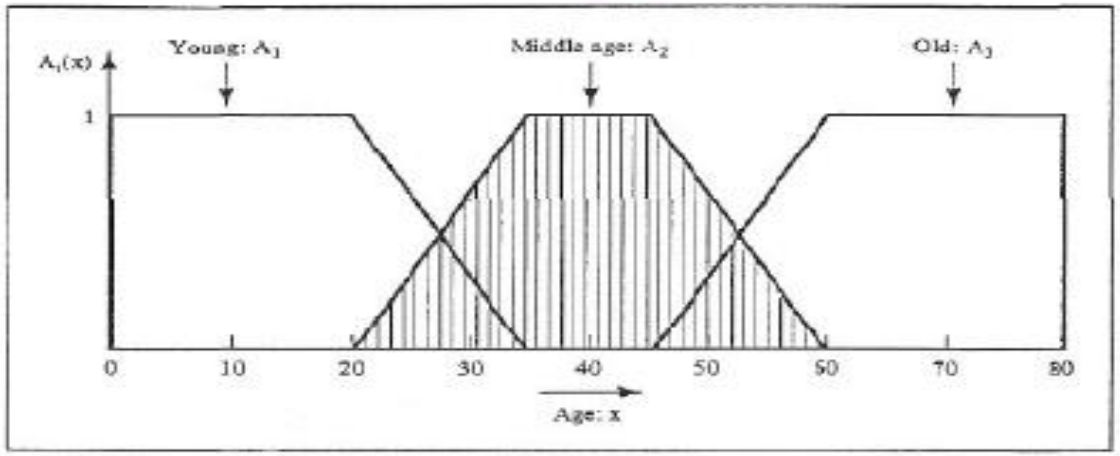
1.4 FUZZY SETS: THEORETIC OPERATIONS INTERSECTION AND UNION

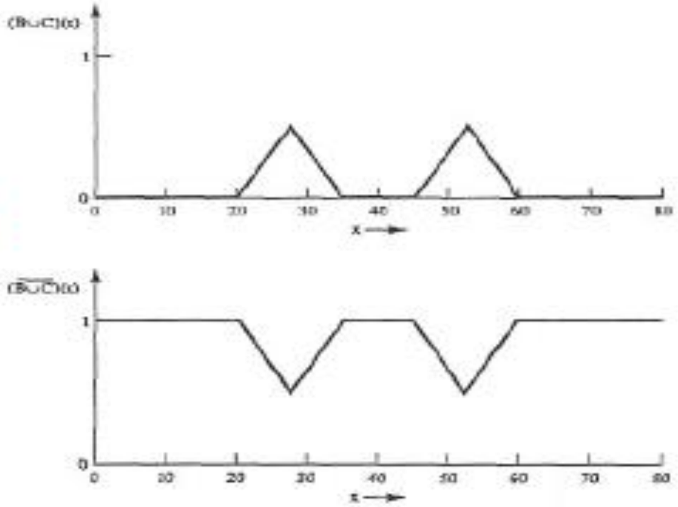
- A_1, A_2, A_3 are normal.
- B and C are subnormal.
- B and C are convex.
- $B \cup C$ and $\overline{B \cup C}$ are not convex.





Normality and convexity may be lost when we operate on fuzzy sets by the standard operations of intersection and complement.





1.4 FUZZY SETS: DISTANCE

- The distance d^p between two sets represented by points in the space is defined as

$$d^{p}(A,B)=\sqrt[p]{\sum_{i=1}^{n}|\mu_A(x_i)-\mu_B(x_i)|^{p}}$$

- If p=2, the distance is the Euclidean distance, if p=1 the distance it is the Hamming distance
- If the point *B* is the empty set (the origin)

$$d^{1}(A,\varphi)=\sum_{i=1}^{n}|\mu_A(x_i)-0|$$

$$d^{1}(A,\varphi)=|A|=\sum_{i=1}^{n}|\mu_A(x_i)|$$

- So, the cardinality of a fuzzy set is the Hamming distance to the origin

1.5 FUZZY SETS: MF FORMULATION (ONE DIMENSIONAL)

Triangular MF:

$$trimf\ (x; a, b, c) = \max\left(\min\left(\frac{x-a}{b-a}, \frac{c-x}{c-b}\right), 0\right)$$

Trapezoidal MF:

$$trapmf\ (x; a, b, c, d) = \max\left(\min\left(\frac{x-a}{b-a}, 1, \frac{d-x}{d-c}\right), 0\right)$$

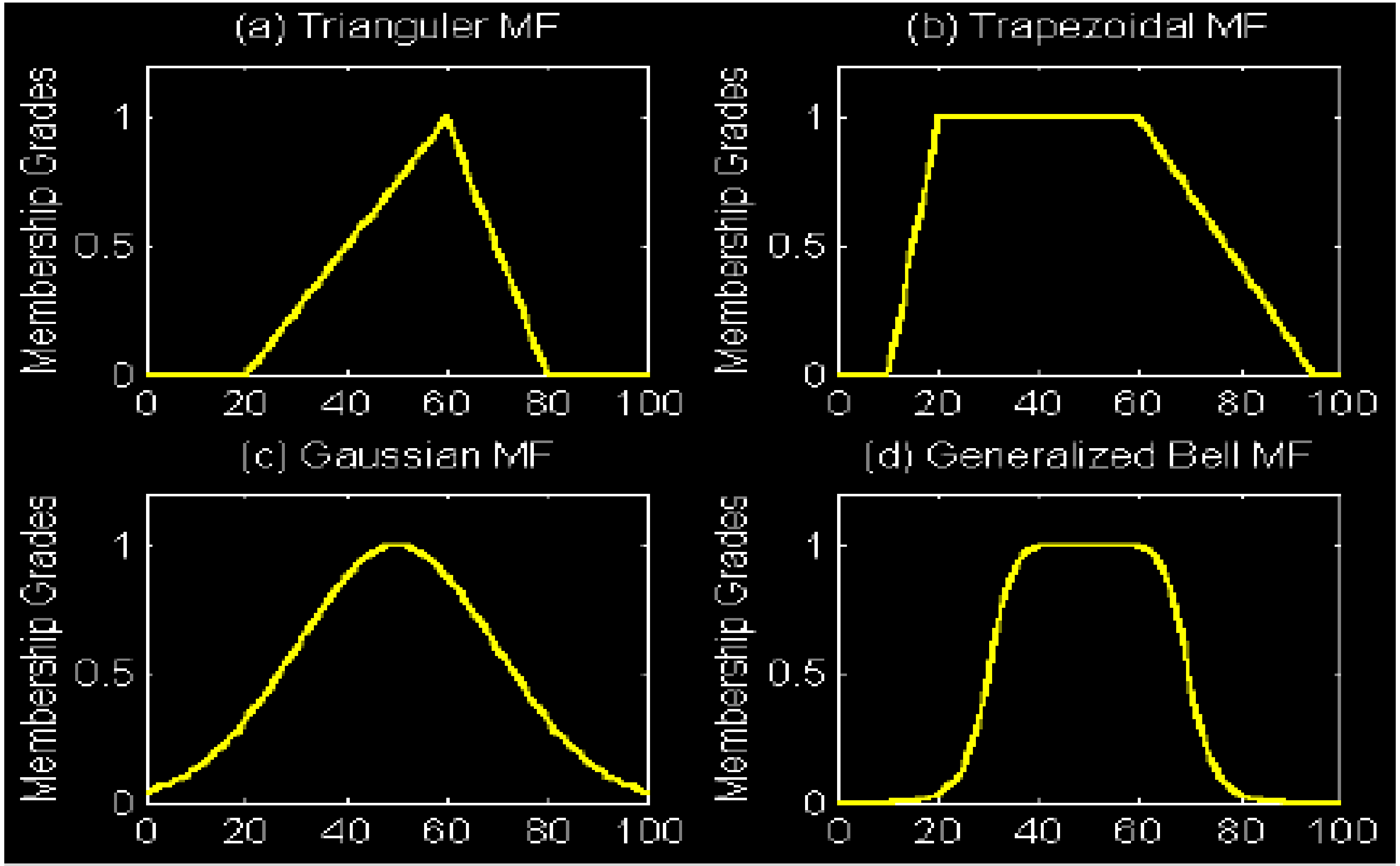
Gaussian MF:

$$gaussmf\ (x; a, c) = e^{-\frac{1}{2}\left(\frac{x-c}{a}\right)^2}$$

Generalized bell MF:

$$gbellmf\ (x; a, b, c) = \frac{1}{1 + \left|\frac{x-c}{a}\right|^{2b}}$$

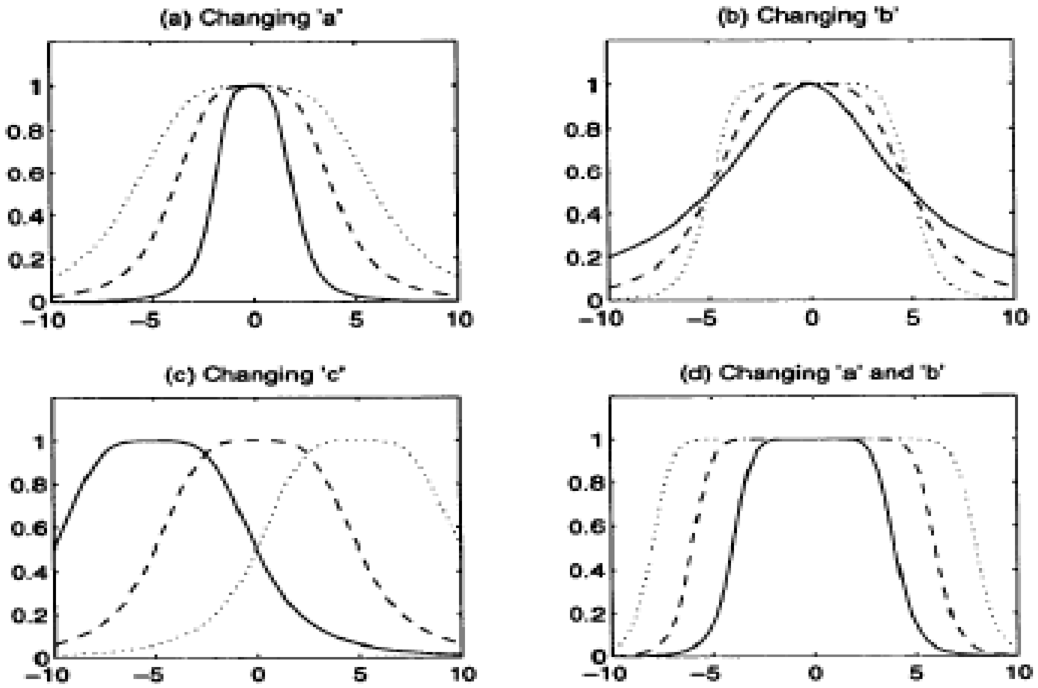
1.5 FUZZY SETS: MF FORMULATION (ONE DIMENSIONAL)



1.5 FUZZY SETS: MF FORMULATION (ONE DIMENSIONAL)

- Generalized Bell MF: $$gbellmf(x; a, b, c) = \frac{1}{1 + \left|\frac{x-c}{a}\right|^{2b}}$$

- Specified by three parameters: {a, b, c}



1.3 FUZZY SETS: HW

Consider three fuzzy sets that represent the concepts of a young, middle-aged, and old person. The membership functions are defined on the interval [0,80] as follows:

$$A_1(x) = \begin{cases} 1 & \text{when } x \le 20 \\ (35 - x)/15 & \text{when } 20 < x < 35 \\ 0 & \text{when } x \ge 35 \end{cases}$$

young

$$A_2(x) = \begin{cases} 0 & \text{when either } x \le 20 \text{ or } \ge 60 \\ (x - 20)/15 & \text{when } 20 < x < 35 \\ (60 - x)/15 & \text{when } 45 < x < 60 \\ 1 & \text{when } 35 \le x \le 45 \end{cases}$$

middle-aged

$$A_3(x) = \begin{cases} 0 & \text{when } x \le 45 \\ (x - 45)/15 & \text{when } 45 < x < 60 \\ 1 & \text{when } x \ge 60 \end{cases}$$

old

1.3 FUZZY SETS: HW

$$A_2(x) = \begin{cases} 0 & \text{when either } x \leq 20 \text{ or } \geq 60 \\ (x-20)/15 & \text{when } 20 < x < 35 \\ (60-x)/15 & \text{when } 45 < x < 60 \\ 1 & \text{when } 35 \leq x \leq 45 \end{cases}$$

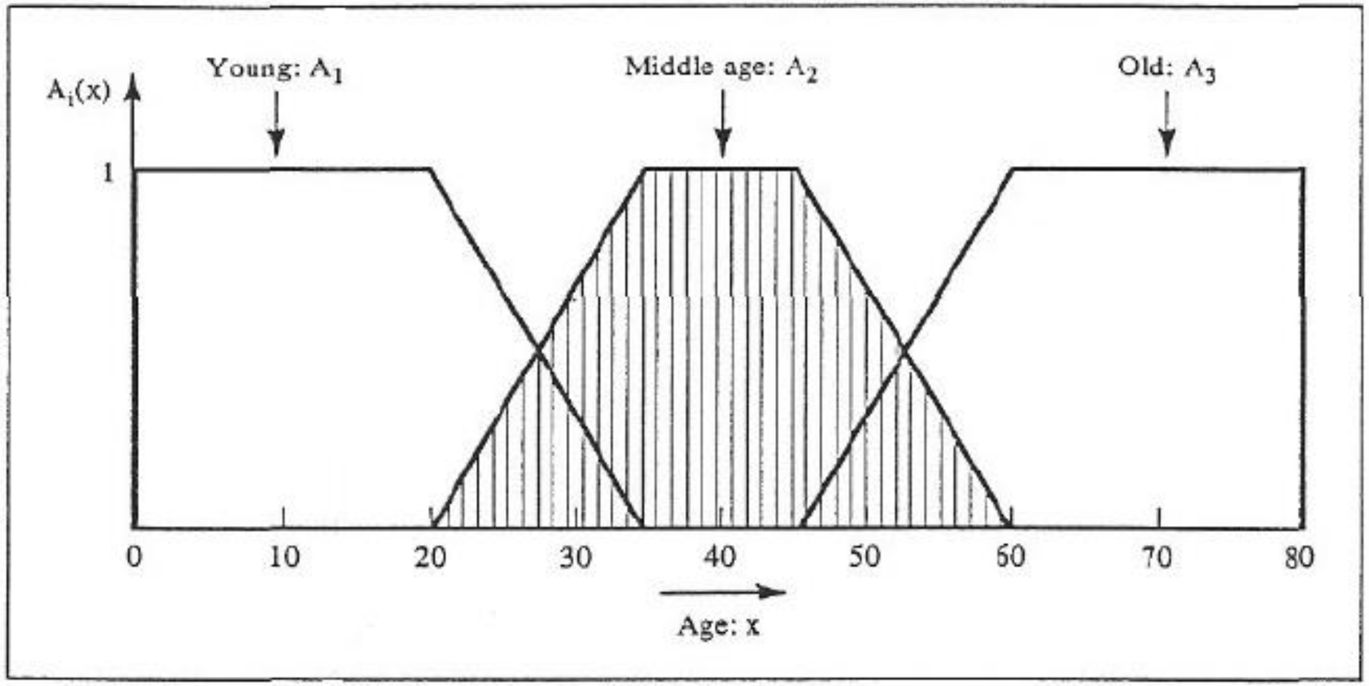


Figure 1.7 Membership functions representing the concepts of a young, middle-aged, and old person. Shown discrete approximation D_2 of A_2 is defined numerically in Table 1.2.

TABLE 1.2 DISCRETE APPROXIMATION OF MEMBERSHIP FUNCTION A_2 (FIG. 1.7) BY FUNCTION D_2 OF THE FORM:
$D_2 : \{0, 2, 4, \ldots, 80\} \to [0, 1]$

x	$D_2(x)$
$x \notin \{22, 24, \ldots, 58\}$	0.00
$x \in \{22, 58\}$	0.13
$x \in \{24, 56\}$	0.27
$x \in \{26, 54\}$	0.40
$x \in \{28, 52\}$	0.53
$x \in \{30, 50\}$	0.67
$x \in \{32, 48\}$	0.80
$x \in \{34, 46\}$	0.93
$x \in \{36, 38, \ldots, 44\}$	1.00

1.3 FUZZY SETS: HW

Find:

- Core of A_2.
- Support of A_2.
- Crossover points of A_2.
- α-cut, strong α –cut of A_2.
- α-cut, strong α –cut of A_2 when α=0.2.
- Scalar cardinality of A_2.
- Are these sets normal?.