FUZZY STATISTICS

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Mohammed
Chapter 1

From Crisp Sets to Fuzzy Sets
1.1 INTRODUCTION

UNCERTAINTY

Uncertainties arise from many sources:

- Random effects
- Measurement errors
- Modeling choices
- Parameter choices
- Inference processes
- Decision making
- Lack of knowledge
1.1 INTRODUCTION

UNCERTAINTY TYPES

Three types of uncertainty are now recognized in the five theories:

- **Nonspecificity** (or imprecision)
  which is connected with sizes (cardinalities) of relevant sets of alternatives.

- **Fuzziness** (or vagueness),
  which results from imprecise boundaries of fuzzy sets.

- **Strife** (or discord)
  which expresses conflicts among the various sets of alternatives.
UNCERTAINTY

FUZZINESS
Lack of definite or sharp distinctions
- vagueness
- cloudiness
- haziness
- unclerness
- indistinctness
- sharplessness

AMBIGUITY
One-to-many relationships

NONSPECIFICITY
Two or more alternatives are left unspecified
- variety
- generality
- diversity
- equivocation
- imprecision

DISCORD
Disagreement in choosing among several alternatives
- dissonance
- incongruity
- discrepancy
- conflict
1.1 INTRODUCTION

Crisp Sets versus Fuzzy Sets

• The crisp set is defined in such a way as to dichotomize the individuals in some given universe of discourse into two groups: members (those that certainly belong in the set) and nonmembers (those that certainly, do not).

• A fuzzy set can be defined mathematically by assigning to each possible individual in the universe of discourse a value representing its grade of membership in the fuzzy set.
1.1 INTRODUCTION

FROM CRISP SETS TO FUZZY SETS

- Probability theory is capable of representing only one of several distinct types of uncertainty.
- When $A$ is a fuzzy set and $x$ is a relevant object, the proposition “$x$ is a member of $A$” is not necessarily either true or false. It may be true only to some degree, the degree to which $x$ is actually a member of $A$. 
1.2 CRISP SETS: AN OVERVIEW

The theory of crisp set

The following general symbols are employed throughout the text:

\[Z = \{\ldots, -2, -1, 0, 1, 2, \ldots\}\] (the set of all integers),
\[N = \{1, 2, 3, \ldots\}\] (the set of all positive integers or natural numbers),
\[N_0 = \{0, 1, 2, \ldots\}\] (the set of all nonnegative integers),
\[N_n = \{1, 2, \ldots, n\},\]
\[N_{0,n} = \{0, 1, \ldots, n\},\]
\[\mathbb{R}: \text{the set of all real numbers},\]
\[\mathbb{R}^+: \text{the set of all nonnegative real numbers},\]
\[[a, b], (a, b], [a, b), (a, b): \text{closed, left-open, right-open, open interval of real numbers between } a \text{ and } b, \text{ respectively},\]
\[(x_1, x_2, \ldots, x_n): \text{ordered } n-\text{tuple of elements } x_1, x_2, \ldots, x_n.\]
1.2 CRISP SETS: AN OVERVIEW

Three basic methods to define sets:

- **The list method**: a set is defined by naming all its members.
  \[ A = \{a_1, a_2, \ldots, a_n\} \]

- **The rule method**: a set is defined by a property satisfied by its members.
  \[ A = \{x \mid P(x)\} \]
  where ‘|’ denotes the phrase “such that”
  \[ P(x) \] a proposition of the form “\( x \) has the property \( P \) ”

- **A set is defined by a characteristic function.**
  \[
  \chi_A(x) = \begin{cases} 
  1 & \text{for } x \in A \\
  0 & \text{for } x \notin A 
  \end{cases}
  \]
  the characteristic function \( \chi_A : X \to \{0,1\} \)
1.2 CRISP SETS: AN OVERVIEW

The union of sets $A$ and $B$:

$$A \cup B = \{ x \mid x \in A \text{ or } x \in B \}$$

The generalized union operation: for a family of sets,

$$\bigcup_{i \in I} A_i = \{ x \mid x \in A_i \text{ for some } i \in I \}$$

The intersection of sets $A$ and $B$:

$$A \cap B = \{ x \mid x \in A \text{ and } x \in B \}$$

The generalized intersection operation: for a family of sets,

$$\bigcap_{i \in I} A_i = \{ x \mid x \in A_i \text{ for all } i \in I \}$$
### 1.2 CRISP SETS: AN OVERVIEW

#### Table 1.1: Fundamental Properties of Crisp Set Operations

<table>
<thead>
<tr>
<th>Property</th>
<th>Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Involution</td>
<td>$\overline{\overline{A}} = A$</td>
</tr>
<tr>
<td>Commutativity</td>
<td>$A \cup B = B \cup A$</td>
</tr>
<tr>
<td></td>
<td>$A \cap B = B \cap A$</td>
</tr>
<tr>
<td>Associativity</td>
<td>$(A \cup B) \cup C = A \cup (B \cup C)$</td>
</tr>
<tr>
<td></td>
<td>$(A \cap B) \cap C = A \cap (B \cap C)$</td>
</tr>
<tr>
<td>Distributivity</td>
<td>$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$</td>
</tr>
<tr>
<td></td>
<td>$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$</td>
</tr>
<tr>
<td>Idempotence</td>
<td>$A \cup A = A$</td>
</tr>
<tr>
<td></td>
<td>$A \cap A = A$</td>
</tr>
<tr>
<td>Absorption</td>
<td>$A \cup (A \cap B) = A$</td>
</tr>
<tr>
<td></td>
<td>$A \cap (A \cup B) = A$</td>
</tr>
<tr>
<td>Absorption by $X$ and $\emptyset$</td>
<td>$A \cup X = X$</td>
</tr>
<tr>
<td></td>
<td>$A \cap \emptyset = \emptyset$</td>
</tr>
<tr>
<td>Identity</td>
<td>$A \cup \emptyset = A$</td>
</tr>
<tr>
<td></td>
<td>$A \cap X = A$</td>
</tr>
<tr>
<td>Law of contradiction</td>
<td>$A \cap \overline{A} = \emptyset$</td>
</tr>
<tr>
<td>Law of excluded middle</td>
<td></td>
</tr>
<tr>
<td>De Morgan’s laws</td>
<td>$A \cap B = A \cup \overline{B}$</td>
</tr>
<tr>
<td></td>
<td>$A \cup B = A \cap \overline{B}$</td>
</tr>
</tbody>
</table>
1.2 CRISP SETS: AN OVERVIEW

Let $R$ denote a set of real number.

- If there is a real number $r$ such that $x \leq r$ for every $x \in R$, then $r$ is called an upper bound of $R$, and $R$ is bounded above by $r$.
- If there is a real number $s$ such that $x \geq s$ for every $x \in R$, then $s$ is called an lower bound of $R$, and $R$ is bounded below by $s$.

For any set of real numbers $R$ that is bounded above, a real number $r$ is called the supremum of $R$ (write $r = \sup R$) iff

(a) $r$ is an upper bound of $R$;

(b) no number less than $r$ is an upper bound of $R$.

For any set of real numbers $R$ that is bounded below, a real number $s$ is called the infimum of $R$ (write $s = \inf R$) iff

(a) $s$ is an lower bound of $R$;

(b) no number greater than $s$ is an lower bound of $R$. 

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1.3 FUZZY SETS: BASIC TYPES

A membership function:

- **A characteristic function**: the values assigned to the elements of the universal set fall within a specified range and indicate the membership grade of these elements in the set.
- Larger values denote higher degrees of set membership.

A set defined by membership functions is a **fuzzy set**.

The most commonly used range of values of membership functions is the **unit interval** \([0,1]\).

We think the universal set \(X\) is always a crisp set.

**Notation:**

- The membership function of a fuzzy set \(A\) is denoted by \(\mu_A\):
  \[\mu_A : X \rightarrow [0,1]\]
- In the other one, the function is denoted by \(A\) and has the same form
  \[A : X \rightarrow [0,1]\]
1.3 FUZZY SETS: BASIC TYPES

Figure 1.2 Examples of membership functions that may be used in different contexts for characterizing fuzzy sets of real numbers close to 2.
1.3 FUZZY SETS: BASIC TYPES

The four fuzzy sets are similar in the sense that the following properties are possessed by each $A_i (i \in \mathbb{N}_4)$:

(i) $A_i(2) = 1$ and $A_i(x) < 1$ for all $x \neq 2$;

(ii) $A_i$ is symmetric with respect to $x = 2$, that is $A_i(2 + x) = A_i(2 - x)$ for all $x \in \mathbb{R}$;

(iii) $A_i(x)$ decreases monotonically from 1 to 0 with the increasing difference $|2 - x|$.

Each function in Fig. 1.2 is a member of a parameterized family of functions.

$$A_1(x) = \begin{cases} p_1(x - r) + 1 & \text{when } x \in [r - 1/p_1, r] \\ p_1(r - x) + 1 & \text{when } x \in [r, r + 1/p_1] \\ 0 & \text{otherwise} \end{cases}$$

$$A_2(x) = \frac{1}{1 + p_2(x - r)^2}$$

$$A_3(x) = e^{-|p_3(x - r)|}$$

$$A_4(x) = \begin{cases} (1 + \cos(p_4 \pi (x - r))) / 2 & \text{when } x \in [r - 1/p_4, r + 1/p_4] \\ 0 & \text{otherwise} \end{cases}$$
1.3 FUZZY SETS: BASIC TYPES

An example:

- Define the seven levels of education:

0 - no education
1 - elementary school
2 - high school
3 - two-year college degree
4 - bachelor’s degree
5 - master’s degree
6 - doctoral degree

![Diagram showing membership functions for different levels of education]

Figure 1.3 Examples of fuzzy sets expressing the concepts of people that are little educated (○), highly educated (●), and very highly educated (□).
1.3 FUZZY SETS: BASIC TYPES

Several fuzzy sets representing linguistic (Value) concepts such as low, medium, high, and so one are often employed to define states of a variable. Such a variable is usually called a fuzzy variable.

For example:

![Diagram showing membership functions for fuzzy sets]

*Figure 1.4* Temperature in the range \([T_1, T_2]\) conceived as: (a) a fuzzy variable; (b) a traditional (crisp) variable.
1.3 FUZZY SETS: MEMBERSHIP FUNCTIONS (MFS)

• Characteristics of MFs:
  ▫ Subjective measures
  ▫ Not probability functions
1.3 FUZZY SETS: MEMBERSHIP FUNCTIONS (MFS)

- **Formal definition:**
  A fuzzy set $A$ in $X$ is expressed as a set of ordered pairs:

$$A = \{(x, \mu_A(x))| x \in X\}$$

Fuzzy set | Membership function (MF) | $X$: Universe or universe of discourse

A fuzzy set is totally characterized by a membership function (MF).
1.3 FUZZY SETS: FUZZY SETS WITH DISCRETE UNIVERSES

• Fuzzy set \( C \) = “desirable city to live in”
  \( X = \{Baghdad, Basra, Erbil\} \) (discrete and non-ordered)
  \( C = \{(Baghdad, 0.1), (Basra, 0.8), (Erbil, 0.9)\} \)

• Fuzzy set \( A \) = “sensible number of children in a family”
  \( X = \{0, 1, 2, 3, 4, 5, 6\} \) (discrete ordered universe)
  \( A = \{(0, .1), (1, .3), (2, .7), (3, 1), (4, .6), (5, .2), (6, .1)\} \)
1.3 FUZZY SETS: FUZZY SETS WITH DISCRETE UNIVERSES

\[ A = \{(0, .1), (1, .3), (2, .7), (3, 1), (4, .6), (5, .2), (6, .1)\} \]
1.3 FUZZY SETS: FUZZY SETS WITH CONT. UNIVERSES

Fuzzy set \( B = \text{“about 50 years old”} \)

\( X = \text{Set of positive real numbers (continuous)} \)

\( B = \{ (x, \mu_B(x)) \mid x \text{ in } X \} \)

\[ \mu_B(x) = \frac{1}{1 + \left( \frac{x - 50}{10} \right)^2} \]
1.3 FUZZY SETS: ALTERNATIVE NOTATION

- A fuzzy set $A$ can be alternatively denoted as follows:

  \[
  \begin{align*}
  &\text{X is discrete} \quad A = \sum_{x_i \in X} \frac{\mu_A(x_i)}{x_i} \\
  &\text{X is continuous} \quad A = \int_X \frac{\mu_A(x)}{x}
  \end{align*}
  \]

  Note that $\sum$ and integral signs stand for the union of membership grades; “/” stands for a marker and does not imply division.
1.3 FUZZY SETS: ALTERNATIVE NOTATION

- **Examples:**

  \[ A = \{(0, 0.1), (1, 0.3), (2, 0.7), (3, 1), (4, 0.7), (5, 0.3), (6, 0.1)\}. \]

  \[ A = 0.1/0 + 0.3/1 + 0.7/2 + 1.0/3 + 0.7/4 + 0.3/5 + 0.1/6, \]

  **A is discrete**

  \[ C = \{(San Francisco, 0.9), (Boston, 0.8), (Los Angeles, 0.6)\}. \]

  \[ C = 0.9/San Francisco + 0.8/Boston + 0.6/Los Angeles, \]

  **C is discrete**

  \[ B = \text{“about 50 years old”} \]

  \[ B = \{(x, \mu_B(x)|x \in X\}, \]

  \[ \mu_B(x) = \frac{1}{1 + \left(\frac{x - 50}{10}\right)^4}. \]

  **B is continuous**

  \[ B = \int_{R^+} \frac{1}{1 + \left(\frac{x - 50}{10}\right)^4} \, dx, \]
1.3 FUZZY SETS: MEMBERSHIP FUNCTIONS?

- Subjective evaluation: The shape of the functions is defined by specialists
- Ad-hoc: choose a simple function that is suitable to solve the problem
- Distributions, probabilities: information extracted from measurements
- Adaptation: testing
- Automatic: algorithms used to define functions from data
Some Definitions

- Support
- Core
- Crossover points (equilibrium points)
- $\alpha$-cut, strong $\alpha$-cut
1.3 FUZZY SETS: MEMBERSHIP FUNCTIONS TERMINOLOGY

Support

- The Support of a fuzzy set $A$ is the set of all points $x$ in $X$ such that:
  \[ \mu_A(x) > 0 \]
- In other words:
  \[ \text{support}(A) = \{x | \mu_A(x) > 0\} \]
1.3 FUZZY SETS: MEMBERSHIP FUNCTIONS TERMINOLOGY

**Core**

- The Core of a fuzzy set \( A \) is the set of all points \( x \) in \( X \) such that:

\[
\mu_A(x) = 1
\]

- In other words:

\[
\text{core}(A) = \{x | \mu_A(x) = 1\}
\]
1.3 FUZZY SETS: MEMBERSHIP FUNCTIONS TERMINOLOGY

Crossover point

- The Crossover point of a fuzzy set $A$ is the set of all points $x$ in $X$ such that:

$$\mu_A(x) = 0.5$$

- In other words:

$$\text{crossover}(A) = \{x|\mu_A(x) = 0.5\}$$
1.3 FUZZY SETS: MEMBERSHIP FUNCTIONS TERMINOLOGY

Fuzzy Singleton

• A fuzzy set whose support is a single point in \( X \) is a fuzz singleton if: \( \mu_A(x) = 1 \)

• Example:
  • A fuzzy singleton
  • “45 years old”
1.3 FUZZY SETS: MEMBERSHIP FUNCTIONS TERMINOLOGY

\( \alpha \)-cut and strong \( \alpha \)-cut

- Given a fuzzy set \( A \) defined on \( X \) and any number \( \alpha \in [0,1] \),
  the \( \alpha \)-cut and strong \( \alpha \)-cut are the crisp sets:

\[
\alpha A = \{ x | A(x) \geq \alpha \} \\
\alpha^+ A = \{ x | A(x) > \alpha \}.
\]

- The \( \alpha \)-cut of a fuzzy set \( A \) is the crisp set that contains all the elements of the universal set \( X \) whose membership grades in \( A \) are greater than or equal to the specified value of \( \alpha \).

- The strong \( \alpha \)-cut of a fuzzy set \( A \) is the crisp set that contains all the elements of the universal set \( X \) whose membership grades in \( A \) are only greater than the specified value of \( \alpha \).
1.3 FUZZY SETS: MEMBERSHIP FUNCTIONS TERMINOLOGY

The height of a fuzzy set $A$:

- The height of a fuzzy set $A$ is the largest membership grade obtained by any element in that set.

$$h(A) = \sup_{x \in X} A(x)$$

- A fuzzy set $A$ is called normal when $h(A) = 1$.
- It is called subnormal when $h(A) < 1$.
- The height of $A$ may also be viewed as the supremum of $\alpha$ for which $\alpha A \neq \phi$. 
1.3 FUZZY SETS: MEMBERSHIP FUNCTIONS TERMINOLOGY

• Scalar cardinality: The cardinality of a fuzzy set is equal to the sum of the membership degrees of all elements.

• The cardinality is represented by $|A|$

$$|A| = \sum_{i=1}^{n} \mu_A(x_i)$$
1.3 FUZZY SETS: CONCAVE FUNCTION

In mathematics, a real-valued function $f$ defined on an interval is called convex, if for any two points $x$ and $y$ in its domain $C$ and any $t$ in $[0,1]$, we have

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y).$$

- In other words, a function is convex if and only if its epigraph (the set of points lying on or above the graph) is a convex set.
1.3 FUZZY SETS: CONVEXITY OF FUZZY SETS

• A fuzzy set \( A \) is convex if for any \( \alpha \) in \([0, 1]\),

\[
\mu_A (\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu_A(x_1), \mu_A(x_2))
\]

Alternatively, \( A \) is convex if all its \( \alpha \)-cuts are convex.

![Diagram](a) Two Convex Fuzzy Sets   (b) A Nonconvex Fuzzy Set

Membership Grades

Membership Grades

0 0.5 1

0 0.5 1
1.4 FUZZY SETS: THEORETIC OPERATIONS

COMPLEMENT

The standard complement of fuzzy set $A$ with respect to the universal set $X$ is defined for all $x \in X$ by the equation $\bar{A}(x) = 1 - A(x)$

Elements of $X$ for which $\bar{A}(x) = A(x)$ are called equilibrium points of $A$.

For example, the equilibrium points of $A_2$ in Fig. 1.7 are 27.5 and 52.5.
1.4 FUZZY SETS: THEORETIC OPERATIONS

(a) Fuzzy Sets A and B

(b) Fuzzy Set "not A"
1.4 FUZZY SETS: THEORETIC OPERATIONS
INTERSECTION AND UNION

Given two fuzzy sets, $A$ and $B$, their intersection and union are defined for all $x \in X$ by the equations

$$(A \cap B)(x) = \min[A(x), B(x)],$$
$$(A \cup B)(x) = \max[A(x), B(x)],$$

$C = A \cap B \iff \mu_c(x) = \min(\mu_A(x), \mu_B(x)) = \mu_A(x) \land \mu_B(x)$

$C = A \cup B \iff \mu_c(x) = \max(\mu_A(x), \mu_B(x)) = \mu_A(x) \lor \mu_B(x)$
1.4 FUZZY SETS: THEORETIC OPERATIONS
INTERSECTION

(a) Fuzzy Sets A and B
(d) Fuzzy Set "A AND B"
1.4 FUZZY SETS: THEORETIC OPERATIONS
UNION
1.4 FUZZY SETS: THEORETIC OPERATIONS
INTERSECTION AND UNION

- $A_1, A_2, A_3$ are normal.
- $B$ and $C$ are subnormal.
- $B$ and $C$ are convex.
- $B \cup C$ and $\overline{B \cup C}$ are not convex.

Normality and convexity may be lost when we operate on fuzzy sets by the standard operations of intersection and complement.
1.4 FUZZY SETS: DISTANCE

- The distance \( d^p \) between two sets represented by points in the space is defined as
\[
d^p (A, B) = \sqrt[p]{\sum_{i=1}^{n} | \mu_A(x_i) - \mu_B(x_i) |^p}
\]

- If \( p=2 \), the distance is the Euclidean distance, if \( p=1 \) the distance it is the Hamming distance.
- If the point \( B \) is the empty set (the origin)
\[
d^1(A, \varphi) = \sum_{i=1}^{n} | \mu_A(x_i) - 0 |
\]
\[
d^1(A, \varphi) = | A | = \sum_{i=1}^{n} | \mu_A(x_i) |
\]
- So, the cardinality of a fuzzy set is the Hamming distance to the origin.
1.5 FUZZY SETS: MF FORMULATION (ONE DIMENSIONAL)

Triangular MF: \[ \text{trimf} \ (x; a, b, c) = \max \left( \min \left( \frac{x - a}{b - a}, \frac{c - x}{c - b} \right), 0 \right) \]

Trapezoidal MF: \[ \text{trapmf} \ (x; a, b, c, d) = \max \left( \min \left( \frac{x - a}{b - a}, 1, \frac{d - x}{d - c} \right), 0 \right) \]

Gaussian MF: \[ \text{gaussmf} \ (x; a, c) = e^{-\frac{1}{2} \left( \frac{x - c}{a} \right)^2} \]

Generalized bell MF: \[ \text{gbellmf} \ (x; a, b, c) = \frac{1}{1 + \left| \frac{x - c}{a} \right|^{2b}} \]
1.5 FUZZY SETS: MF FORMULATION (ONE DIMENSIONAL)
1.5 FUZZY SETS: MF FORMULATION (ONE DIMENSIONAL)

- Generalized Bell MF: \[ gbellmf \ (x; a, b, c) = \frac{1}{1 + \left( \frac{x - c}{a} \right)^{2b}} \]

- Specified by three parameters: \{a, b, c\}
1.3 FUZZY SETS: HW

Consider three fuzzy sets that represent the concepts of a young, middle-aged, and old person. The membership functions are defined on the interval [0,80] as follows:

$$A_1(x) = \begin{cases} 
1 & \text{when } x \leq 20 \\
(35 - x)/15 & \text{when } 20 < x < 35 \\
0 & \text{when } x \geq 35
\end{cases}$$

$$A_2(x) = \begin{cases} 
0 & \text{when either } x \leq 20 \text{ or } \geq 60 \\
(x - 20)/15 & \text{when } 20 < x < 35 \\
(60 - x)/15 & \text{when } 45 < x < 60 \\
1 & \text{when } 35 \leq x \leq 45
\end{cases}$$

$$A_3(x) = \begin{cases} 
0 & \text{when } x \leq 45 \\
(x - 45)/15 & \text{when } 45 < x < 60 \\
1 & \text{when } x \geq 60
\end{cases}$$
1.3 FUZZY SETS: HW

\[ A_2(x) = \begin{cases} 
0 & \text{when either } x \leq 20 \text{ or } \geq 60 \\
(x - 20)/15 & \text{when } 20 < x < 35 \\
(60 - x)/15 & \text{when } 45 < x < 60 \\
1 & \text{when } 35 \leq x \leq 45 
\end{cases} \]

**Figure 1.7** Membership functions representing the concepts of a young, middle-aged, and old person. Shown discrete approximation \( D_2 \) of \( A_2 \) is defined numerically in Table 1.2.

**Table 1.2** Discrete Approximation of Membership Function \( A_2 \) (Fig. 1.7) by Function \( D_2 \) of the Form: \( D_2 : [0, 2, 4, \ldots, 80] \to [0, 1] \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( D_2(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x \notin {22, 24, \ldots, 58} )</td>
<td>0.00</td>
</tr>
<tr>
<td>( x \in {22, 24, \ldots, 58} )</td>
<td>1.00</td>
</tr>
</tbody>
</table>

1.3 FUZZY SETS: HW

Find:

- Core of $A_2$.
- Support of $A_2$.
- Crossover points of $A_2$.
- $\alpha$-cut, strong $\alpha$-cut of $A_2$.
- $\alpha$-cut, strong $\alpha$-cut of $A_2$ when $\alpha=0.2$.
- Scalar cardinality of $A_2$.
- Are these sets normal?